Proof mining and fixed points of generalized p-contractive mappings on metric spaces

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Introduction

There exist fixed point theorems for mappings on metric spaces underlying conditions reminiscent of but weaker than contractivity.

We derive improved versions of some such theorems with the help of insights and techniques from proof mining. General references on proof mining will be work by Kohlenbach and various co-authors, e.g. Gerhardy, Lambov, Leuştean and Oliva.

History and general setting

First contraction condition (Banach, 1922):

(1) A function $f : X \to X$ on a metric space (X, d) is a *contraction* if there exists $a < 1$ such that

$$\forall x, y \in X(d(f(x), f(y)) \leq ad(x, y)).$$

Theorem. *If (X, d) is a complete metric space and $f : X \to X$ a contraction, then f has a unique fixed point z, and for all $x_0 \in X$*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

(3) A function $f : X \to X$ on a metric space (X, d) is *contractive* if

$$\forall x, y \in X (x \neq y \to d(f(x), f(y)) < d(x, y)).$$

Classic result due to Edelstein.

Theorem. (Edelstein, 1962) *Let (X, d) be a* compact *metric space, and let $f : X \to X$ be contractive. Then f has a unique fixed point z, and for all $x_0 \in X$*

$$\lim_{n \to \infty} f^n(x_0) = z.$$

This has led to the study of a range of weakened or alternative contraction conditions. The hope is then to derive corresponding generalizations of the fixed point theorems.

R. Kannan (1968) proved a similar theorem for complete metric spaces (X, d) and $f : X \to X$ satisfying:

(4) There exists $a < 1/2$ such that for all $x, y \in X$

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))].$$

In contrast to mappings satisfying (1) or (3) these need not be continuous.

Systematization of contractivity conditions by B. E. Rhoades in the seventies:

25 basic conditions (1) - (25) considered in the setting of complete metric spaces. Also standard generalizations of each of the (25) conditions are treated, resulting in a large number of different contractivity conditions. (We have seen (1), (3) and (4).)

These are compared, with implications:

Any function which satisfies (1) satisfies (3).

The comparison was completed in 1997 by P. Collaço and J. Carvalho e Silva.

Condition (25) is the most general. Any function satisfying one of (1) - (24) also satisfies (25). A fixed point theorem for (25) would entail fixed point theorems for (1) - (24). Condition (25) is:

(25) $\forall x, y \in X(x \neq y \to d(f(x), f(y)) < \text{diam}\{x, y, f(x), f(y)\}).$

A function $f : X \to X$ is nonexpansive if:

$$\forall x, y \in X(d(f(x), f(y)) \leq d(x, y)).$$

A function f satisfying (25) is not necessarily nonexpansive.

However:

There does not necessarily exist a fixed point for functions on complete metric spaces satisfying (25).

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x + 1$. Then f satisfies (25).

If we restrict ourselves to the case where (X, d) is compact and f is continuous this changes. We can also consider the following standard generalization of (25):

(50) There exists a $p \in \mathbb{N}$ such that

$$\forall x, y \in X (x \neq y \rightarrow d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}).$$

Here $f^p(x)$ stands for f iterated p times, with starting point x. A function satisfying (50) we call *generalized* p*-contractive*.

The original theorem

J. Kincses and V. Totik proved:

Theorem. (Kincses and Totik, 1990) *Let* (X, d) *be a compact metric space, and let* $p \in \mathbb{N}$. *Let* $f : X \to X$ *be continuous and generalized* p*-contractive. Then* f *has a unique fixed point* z*, and for every* $x_0 \in X$ *we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

This theorem does not extend to the other standard generalizations, e.g.:

(75) There exist $p, q \in \mathbb{N}$ such that

$$\forall x, y \in X (x \neq y \to d(f^p(x), f^q(y)) < \operatorname{diam}\{x, y, f^p(x), f^q(y)\}).$$

(125) There exists $p : X \times X \to \mathbb{N}$ such that

$$\forall x, y \in X (x \neq y \to d(f^{p(x,y)}(x), f^{p(x,y)}(y)) < \operatorname{diam}\{x, y, f^{p(x,y)}(x), f^{p(x,y)}(y)\}).$$

Techniques and insights from proof mining are applicable to find an improved version of this theorem. We present a qualitatively improved result which also supply numerical information on the convergence.

We need a number of definitions.

Definition 1. *Let* (X, d) *be a metric space, let* $p \in \mathbb{N}$ *and let* $f : X \to X$*. We say that* f *is* uniformly generalized p-contractive *if for all real* $\varepsilon > 0$ *there exists* $\delta \in \mathbb{R}$ *such that*

$$\forall x, y \in X (d(x, y) > \varepsilon \to$$
$$\mathrm{diam}\{x, y, f^p(x), f^p(y)\} - d(f^p(x), f^p(y)) > \delta).$$

Modulus of uniform generalized p-contractivity

The following is just a way of rephrasing the statement that f is uniformly generalized p-contractive.

Definition 2. *Let* (X, d) *be a metric space, and let* $f : X \to X$*. We say that* $\eta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ *is a* modulus of uniform generalized p-contractivity *for* f *if* η *satisfies*

$$\forall x, y \in X \forall \delta \in \mathbb{Q}^*_+ (d(x, y) > \delta \to$$

$$d(f^p(x), f^p(y)) + \eta(\delta) < diam\{x, y, f^p(x), f^p(y)\}).$$

Modulus of uniform continuity

In the same way we can express that a mapping f on a metric space is uniformly continuous by saying that it has a modulus of uniform continuity.

Definition 3. *Let* (X, d) *be a metric space, and let* $f : X \to X$*. We say that* $\omega : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ *is a* modulus of uniform continuity *for* f *if* ω *satisfies*

$$\forall x, y \in X \forall \delta \in \mathbb{Q}^*_+ (d(x, y) < \omega(\delta) \to d(f(x), f(y)) < \delta).$$

Remark 1. *Let* (X, d) *be a compact metric space, and let* $f : X \to X$ *be continuous and generalized* p*-contractive. Then* f *has moduli* ω *and* η *of uniform continuity and uniform generalized* p*-contractivity.*

Rate of convergence

The quantitative version of Kincses and Totik's Theorem will involve a rate of convergence.

Definition 4. *Let* (X, d) *be a metric space, and let* (x_n) *be a sequence in* X *converging to* z*. We say that* $\phi : \mathbb{Q}^*_+ \to \mathbb{N}$ *is a* rate of convergence *for* (x_n) *if*

$$\forall \delta \in \mathbb{Q}^*_+ \forall n \geq \phi(\delta)(d(z, x_n) \leq \delta).$$

If

$$\forall x_0 \in X \forall \delta \in \mathbb{Q}^*_+ \forall n \geq \phi(\delta)(d(z, x_n) \leq \delta),$$

where (x_n) *is defined by* $x_{n+1} := f(x_n)$*, then we say that* ϕ *is a* rate of convergence *for the fixed point of* f*.*

Cauchy rate

Since we will consider cases where the space is not complete, we include the following.

Definition 5. *Let* (X, d) *be a metric space, and let* (x_n) *be a sequence in* X*. We say that* $\rho : \mathbb{Q}^*_+ \to \mathbb{N}$ *is a* Cauchy rate *for* (x_n) *if*

$$\forall \delta \in \mathbb{Q}^*_+ \forall m, n \geq \rho(\delta)(d(x_m, x_n) \leq \delta).$$

If

$$\forall x_0 \in X \forall \delta \in \mathbb{Q}^*_+ \forall m, n \geq \rho(\delta)(d(x_m, x_n) \leq \delta),$$

where (x_n) *is defined by* $x_{n+1} := f(x_n)$*, then we say that* ρ *is a* Cauchy rate *for* f*.*

Theorem 1. *Let (X, d) be a metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus ω of uniform continuity, and a modulus η of uniform generalized p-contractivity. Let $x_0 \in X$ be the starting point of a sequence (x_n) defined by $x_{n+1} := f(x_n)$. Suppose (x_n) is bounded, and let b be a bound on d when restricted to (x_n). Let a be a rational number such that $0 < a < 1$. Let $\rho : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ be defined by*

$$\rho(\delta) := \min\{\eta(\delta), \frac{\delta}{2}, \eta(a\omega^p(\frac{\delta}{2}))\}.$$

Let $\phi : \mathbb{Q}^*_+ \to \mathbb{N}$ be defined by

$$\phi(\delta) := \begin{cases} p\,\lceil (b - \delta)/\rho(\delta) \rceil & \textit{if } b > \delta, \\ 0 & \textit{otherwise.} \end{cases}$$

Then ϕ is a Cauchy rate for (x_n).

Aspects of Theorem 1

The following is explained by logical metatheorems:

1. We do not require the space to be compact. Nor even separable. What we require is only a bound b on the iteration sequence. Without completeness we have a Cauchy rate. (If the space is complete, then the Cauchy rate is of course a rate of convergence.)

2. We have independence of the Cauchy rate from the function, the space, and the starting point, as long as the bound b and the moduli η and ω stay fixed. In particular, if b is a bound on the whole space, then the Cauchy rate is independent of the starting point.

But the following is not explained:

That we get a rate of convergence (computable in the moduli) instead of a rate of maximum proximity (computable in the moduli). (The space for simplicity here assumed complete.)

Definition 6. *Let* (X, d) *be a metric space, and let* $f : X \to X$*. Let* (x_n) *be a sequence in* (X, d)*. Let* $z \in X$*. We say that* $\psi : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$ *is a* rate of maximum proximity *for* (x_n) *to* z *if* ψ *satisfies*

$$\forall \delta \in \mathbb{Q}_+^* \exists n \leq \psi(\delta)(d(x_n, z) < \delta).$$

If we have

$$\forall x_0 \in X \forall \delta \in \mathbb{Q}_+^* \exists n \leq \psi(\delta)(d(x_n, z) < \delta),$$

where (x_n) *is defined by* $x_{n+1} := f(x_n)$*, then we say that* ψ *is a* rate of maximum proximity *for* f *to* z*.*

Corollary 1. *Let (X, d) be a bounded complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be uniformly continuous and uniformly generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

Corollary 2. (Kincses and Totik's Theorem) *Let (X, d) be a compact metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be continuous and generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

Modulus of uniqueness

Proposition 1. *Let* (X, d) *be a metric space. Let* $f : X \to X$ *have modulus* η *of uniform generalized p-contractivity. Define* $\psi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ *by* $\psi(\delta) := \eta(\delta)/2$*. Then* ψ *satisfies*

$$\forall y_1, y_2 \in X \forall \delta \in \mathbb{Q}^*_+ (\bigwedge_{i=1}^{2} d(y_i, f^p(y_i)) < \psi(\delta) \to d(y_1, y_2) < \delta).$$

We express this by saying that f^p has what in the context of proof mining has been called a modulus of uniqueness. (Notion in full generality introduced by Kohlenbach, 1993.)

Theorem 2. *Let* (X, d) *be a metric space, and let* $f : X \to X$ *have a modulus* η *of uniform generalized* p*-contractivity and a modulus* ω *of uniform continuity. For* $x_0 \in X$ *define the iteration sequence* (x_n) *by* $x_{n+1} := f(x_n)$*. Suppose for some* $x_0 \in X$ *the iteration sequence is bounded. Then for every choice of* $x_0 \in X$ *the iteration sequence* (x_n) *is bounded.*

Proposition 2. *Let* (X, d) *be a metric space, and let* $f : X \to X$ *have moduli* ω *of uniform continuity and* η *of uniform generalized* p*-contractivity. Let* (x_n) *be a Picard iteration sequence bounded by* $b \in \mathbb{R}$*. We cannot in the general case find a Cauchy rate for* (x_n) *expressed by only two of* ω*,* η *and* b*.*

Asymptotic contractions

For uniformly generalized p-contractive functions we could get a rate of convergence even though only a rate of maximum proximity to the fixed point was assured. We have the same situation for asymptotic contractions.

Asymptotic contractions were introduced by W. A. Kirk in 2003.

Definition 7. *A function* $f : X \to X$ *on a metric space* (X, d) *is called an* asymptotic contraction *with moduli* $\phi, \phi_n : [0, \infty) \to [0, \infty)$ *if* ϕ, ϕ_n *are continuous,* $\phi(s) < s$ *for all* $s > 0$ *and for all* $x, y \in X$

$$d(f^n(x), f^n(y)) \leq \phi_n(d(x, y)),$$

and moreover $\phi_n \to \phi$ *uniformly on the range of* d.

Kirk also proved a fixed point theorem stating that given a complete metric space (X, d) and a continuous asymptotic contraction f, if for some $x \in X$ the Picard iteration sequence $(f^n(x))_{n\in\mathbb{N}}$ is bounded, then for every starting point $x \in X$ the iteration sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to the unique fixed point of f.

P. Gerhardy has with the help of proof mining given an elementary proof of this theorem, and in the process also given a rate of maximum proximity for such a $(f^n(x))_{n\in\mathbb{N}}$ to the fixed point (expressed in relevant moduli and a bound on the sequence). This involves giving a new definition of asymptotic contractions. The new definition covers the old.

Definition 8. (Gerhardy) *A function* $f : X \to X$ *on a metric space* (X, d) *is called an* asymptotic contraction *if for each* $b > 0$ *there exists moduli* $\eta^b : (0, b] \to (0, 1)$ *and* $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ *and the following hold:*

1. *There exists a sequence of functions* $\phi_n^b : (0, \infty) \to (0, \infty)$ *such that for all* $x, y \in X$*, for all* $\varepsilon > 0$ *and for all* $n \in \mathbb{N}$

$$b \geq d(x, y) \geq \varepsilon \;\; \text{gives} \;\; d(f^n(x), f^n(y)) \leq \phi_n^b(\varepsilon) d(x, y).$$

2. *For each* $0 < l \leq b$ *the function* $\beta_l^b := \beta^b(l, \cdot)$ *is a modulus of uniform convergence for* ϕ_n^b *on* $[l, b]$*, i.e.*

$$\forall \varepsilon > 0 \forall s \in [l, b] \forall m, n \geq \beta_l^b(\varepsilon)(|\phi_m^b(s) - \phi_n^b(s)| \leq \varepsilon).$$

3. *Define* $\phi^b := \lim_{n \to \infty} \phi_n^b$*. Then for each* $0 < \varepsilon \leq b$ *we have*

$$\phi^b(s) + \eta^b(\varepsilon) \leq 1$$

for each $s \in [\varepsilon, b]$*.*

The following theorem is due to Gerhardy, except that it is numerically improved.

Theorem 3. *Let* (X, d) *be a complete metric space, let* f *be a continuous asymptotic contraction and let* $b > 0$ *and* η, β *be given. If for some* $x_0 \in X$ *the Picard iteration sequence* $(x_n)_{n\in\mathbb{N}}$ *is bounded by* b *then* f *has a unique fixed point* z*,* $(x_n)_{n\in\mathbb{N}}$ *converges to* z *and for every* $\varepsilon > 0$ *there exists an* $m \leq M_\varepsilon$ *such that*

$$d(x_m, z) \leq \varepsilon,$$

where

$$M_\varepsilon(\eta, \beta, b) := k \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg(1 - \frac{\eta(\varepsilon)}{2})} \right\rceil,$$

with $k := \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$.

Theorem 4. *Let (X, d) be a complete metric space, let $b > 0$ be given, and let f be a continuous asymptotic contraction with moduli η and β. If for some $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded by b then $(x_n)_{n\in\mathbb{N}}$ has the following rate of convergence. Let z be the unique fixed point. Let $\varepsilon > 0$ and let $n \in \mathbb{N}$ satisfy*

$$\begin{aligned} n \geq \max\{&k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1), \\ &(k-1) \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + M_\gamma + 1\}, \end{aligned}$$

where

$k := \left\lceil \frac{\lg(\varepsilon)-\lg(b)}{\lg(1-\frac{\eta(\gamma)}{2})} \right\rceil$, $M_\gamma := K_\gamma \cdot \left\lceil \frac{\lg(\gamma)-\lg(b)}{\lg(1-\frac{\eta(\gamma)}{2})} \right\rceil$, $K_\gamma := \beta_\gamma(\frac{\eta(\gamma)}{2})$, *and* $\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}$, $\gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}$. *Then*

$$d(x_n, z) \leq \varepsilon.$$

Final comments

The above theorem gives a rate of convergence for appropriate sequences, instead of a rate of maximum proximity. This is again not explained by the logical metatheorems.

We restrict the discussion to the case where the space is bounded. Having a theorem expressing the convergence of a Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$ to a unique fixed point, one can consider the following:

(i) $\forall x, y \in X(f(x) = x \wedge f(y) = y \rightarrow x = y)$,

(ii) $\forall x_0 \in X \forall \delta \in \mathbb{Q}^*_+ \exists n(d(x_n, x_{n+1}) < \delta)$.

With X and $\mathbb{R}$ suitably represented (i) and (ii) can be written in the form

$$\forall m^0 \forall x^X \exists n^0 A_{qf}(m, x, n).$$

Given a suitable proof one can with the help of proof mining find from (i) a modulus of uniqueness for f, and from (ii) a modulus ψ which gives a bound on how far one has to go in the iteration to at least once have $d(x_n, x_{n+1}) < \delta$, independent of the starting point x_0. These two moduli can be combined to provide a rate of maximum proximity for f to the fixed point.

However, if we want a modulus which can be combined with a modulus of uniqueness to provide a rate of convergence, considering (ii) will not do. The following statement,

(iii) $\forall x_0 \in X \forall \delta \in \mathbb{Q}^*_+ \exists n \forall m \geq n(d(x_{m+1}, x_m) \leq \delta)$,

would have provided a modulus ψ' such that

$$\forall x_0 \in X \forall \delta \in \mathbb{Q}^*_+ \forall m \geq \psi'(\delta)(d(x_{m+1}, x_m) \leq \delta),$$

which is what we need. But (iii) has the logical form

$$\forall m^0 \forall x^X \exists n^0 \forall k^0 A_{qf}(m, x, n, k),$$

which is too complex if the proof uses classical logic.

We have shown two cases from "real world" mathematics where convergence to the fixed point is not monotone, but where we still find rates of convergence (computable in the moduli) instead of rates of maximum proximity (computable in the moduli). We cannot do this in the general case. But can we say something interesting about when this is possible?