Computer Algebra Proofs for Combinatorial Inequalities and Identities

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- Routines are desired which not only *prove* but also *find* such identities.

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- "Combinatorial" here just means that the inequality depends on a discrete parameter n. Inequalities like $\sin x < x$ $(x \geq 0)$ are out of scope.

Proving Combinatorial Identities

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- *Today:* An algorithm for proving identities, which is applicable to a much larger input class.

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- Proof: If N has this property and $f_0 = \cdots = f_{N-1} = 0$ then $f \equiv 0$ by induction. If not $f_0 = \cdots = f_{N-1} = 0$, then $f \not\equiv 0$ anyway.

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- Here, we need not assume that $x_0, \dots, x_N$ be independent! If (f_n) is defined via recurrence equations, then these equations give rise to known polynomial relations

$$p_1(x_0, \dots, x_N) = \cdots = p_m(x_0, \dots, x_N) = 0$$

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- This can be decided using *Gröbner Bases.* $\square$

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- The proof is completed by checking the claim for $n = 0$.

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h_n	h_{n+1}	h_{n+2}	h_{n+3}	h_{n+4}	$\cdots$
f_n	f_{n+1}	f_{n+2}	f_{n+3}	f_{n+4}	$\cdots$

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- In particular: *Zero equivalence is decidable for this class.*

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Proving Combinatorial Inequalities

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- *Today:* A method for proving inequalities, which succeeds for a great many instances.

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  does the job.
- Proof: If N has this property and $f_0 > 0, \dots, f_{N-1} > 0$ then $f > 0$ by induction. If not $f_0 > 0, \dots, f_{N-1} > 0$, then $f \not> 0$ anyway.

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- This knowledge may be anything that gives rise to polynomial (in)equalities for the x_i.

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- This can be decided using *Cylindrical Algebraic Decomposition.* $\square$
- The method can be applied to the same class of sequences as the identity prover explained before.

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- The proof is completed by checking the claim for $n = 0$.

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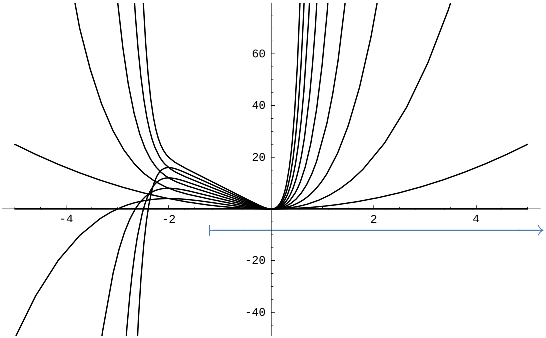
- Let's have a look at the functions $(z+1)^n - (1+nz)$ for $n = 1, 2, 3, \ldots$:



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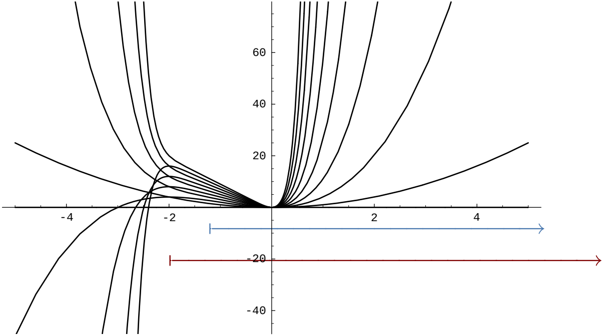

- The picture suggests that Bernoulli's inequality already holds for $z \geq -2$. Is this true?

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- Conclusion: We have generalized Bernoulli's inequality.

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- $P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0$ (for $-1 \leq x \leq 1$)

Some further Examples

- $$\prod_{k=0}^{n} \frac{3k+4}{3k+2} > 1 + \frac{2}{3} \sum_{k=1}^{n+1} \frac{1}{k}$$
- $$\sum_{k=1}^{n} \frac{(2F_{k+1} - F_k)^2}{F_k} \geq \frac{(3F_{n+1} + F_n - 3)^2}{F_{n+2} - 1}$$
- $$\Bigl(\sum_{k=1}^{n} x_k y_k\Bigr)^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2$$
- $$\prod_{k=1}^{n-1} (a^k + 1) < \frac{1-a}{a^n - 2a + 1} \text{ (for } 0 < a < \tfrac{1}{2}\text{)}$$
- $P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0$ (for $-1 \leq x \leq 1$)
- ...

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  - It's just a method that often succeeds.