Proof Mining:
Applications of Proof Theory to Analysis I

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New results by logical analysis of proofs

Input: Ineffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,
- generalizations of proofs: weakening of premises.

Logical methods I:

Elimination of detours (no lemmas): direct proofs

- Extraction and subsequent analysis of Herbrand terms (Herbrand 1930): used e.g. in H. Luckhardt's analysis of a proof of Roth's theorem (first polynomial bounds on number of solutions; also by Bombieri/van der Poorten).
- ε-term elimination (D. Hilbert, W. Ackermann, G. Mints): used in C. Delzell's effective versions of the 17th Hilbert problem.
- Cut-elimination (G. Gentzen, 1936): used in J.-Y. Girard's analysis of Van der Waerden's theorem and by A. Weiermann in combinatorics.

Limitations

- Techniques work only for restricted formal contexts: mainly purely universal ('algebraic') axioms, restricted use of induction, no higher analytical principles.
- Require that one can 'guess' the correct Herbrand terms: in general procedure results in proofs of length $2_n^{|P|}$, where $2_{n+1}^k = 2^{2_n^k}$ (n cut complexity).

Logical methods II: Proof Interpretations

- **interpret** the formulas A occurring in the proof $P : A \mapsto A^I$,
- interpretation C^I of the conclusion contains the **additional information** searched for,
- construct by **recursion on P** a new proof P^I of C^I.

Modus Ponens Problem: $\dfrac{A^I, \quad (A \to B)^I}{B^I}$.

Special case of the Modus Ponens-Problem

$$\frac{A :\equiv \forall x\, \exists y\, \forall z\, A_{qf}(x,y,z) \qquad \forall x\, \exists y\, \forall z\, A_{qf}(x,y,z) \to \forall u\, \exists v\, B_{qf}(u,v)}{\forall u\, \exists v\, B_{qf}(u,v)}.$$

1. Attempt: Explicit realization of existential quantifiers:

$$\frac{\forall x,z\, A_{qf}(x,\varphi(x),z) \qquad \forall f\Big(\forall x,z\, A_{qf}(x,f(x),z) \to \forall u\, B_{qf}(u,\Phi(u,f))\Big)}{\forall u\, B_{qf}(u,\Phi(u,\varphi))}.$$

Discussion

- works for intuitionistic proofs ('m-realizability').
- for **classical proofs** of A: i.g. no computable φ!

Examples

1) $P(x,y)$ decidable, but $Q(x) := \exists y\, P(x,y)$ undecidable.

$$\forall x \exists y \forall z (P(x,y) \vee \neg P(x,z))$$

logically true, but **no computable** φ $(x,y,z \in \mathbb{N})$.

2) $(a_n)_{n\in\mathbb{N}}$ nonincreasing sequence in $[0,1]\cap\mathbb{Q}$. Then

$$\text{PCM}(a_n) :\equiv \forall x \exists y \forall z \geq y (|a_y - a_z| \leq 2^{-x}).$$

Even for prim.rec. (a_n) i.g. **no computable bound** for y (Specker 1947).

2. Attempt: Gödel's functional interpretation (1958)

$\forall x \exists y \forall z\, A_{qf}(x,y,z)$ classically provable $\overset{\text{Gödel(33)}}{\Rightarrow}$

$\forall x \neg\neg \exists y \forall z\, A_{qf}(x,y,z)$ intuitionistically provable $\Rightarrow$

$\forall x, g \exists y\, A_{qf}(x,y,g(y))$ semi-intuitionistically provable.

Consider

$$\forall x, g A_{qf}(x, \Phi(x,g), g(\Phi(x,g)))$$

(**no-counterexample interpretation**)

Again: Modus Ponens

$$\begin{cases} \forall x, g A_{qf}(x, \Phi(x,g), g(\Phi(x,g))), \\ \forall u, Y (\forall x, g (A_{qf}(x, Y(x,g), g(Y(x,g))) \to B_{qf}(u, \Omega(u,Y))). \end{cases}$$

Then: $\forall u\, B_{qf}(u, \Omega(u, \Phi))$.

Examples:

1) Define $\Phi(x,g) := \begin{cases} x, & \text{if } \neg P(x, g(x)) \\ g(x), & \text{otherwise.} \end{cases}$

2) $\Phi((a_n), x, g) :=$

$$\min y \leq \max_{i \leq 2^x} (g^i(0)) [g(y) \geq y \to |a_y - a_{g(y)}| \leq 2^{-x}].$$

3. Attempt: Monotone functional interpretation (K.96)

Definition 1 (Howard) *(x^* majorizes x):*

$$x^* \; maj_0 \; x :\equiv x^* \geq x,$$

$$x^* \; maj_{\rho\to\tau} \; x :\equiv \forall y^*, y(y^* \; maj_\rho y \to x^* y^* \; maj_\tau \; xy).$$

Extract Φ^*, Ω^* with

$$\exists\Phi\Big(\Phi^* \; maj \; \Phi \wedge \forall x, g A_{qf}(x, \Phi(x,g), g(\Phi(x,g)))\Big) \text{ and}$$
$$\exists\Omega\Big(\Omega^* \; maj \; \Omega \wedge \forall u, Y(\ldots \to B_{qf}(u, \Omega(u,Y)))\Big).$$

Define $F^*(u) := \Omega^*(u, \Phi^*)$. Then

$$\forall u \exists v \leq F^*(u) \, B_{qf}(u,v).$$

Examples

1) $\Phi(x,g) := \begin{cases} x, & \text{if } \neg P(x,g(x)) \\ g(x), & \text{otherwise.} \end{cases}$

Put: $\Phi^*(x,g) := \max(x, g(x))$ **independence from P!**

2) $\Phi((a_n), x, g) :=$

$$\min y \leq \max_{i \leq 2^x}(g^i(0))[g(y) \geq y \to |a_y - a_{g(y)}| \leq 2^{-x}].$$

Put: $\Phi^*((a_n), x, g) := \max_{i \leq 2^x}(g^i(0))$ **independence from (a_n)!**

Extraction algorithm by MFI: **cubic complexity**

(M.-D.Hernest/K.,TCS 2005).

Other uses of proof interpetations

- Combinations of negative and Friedman/Dragalin translation with modified realizability (Berger/Buchholz/Schwichtenberg 2002, Coquand/Hofmann 1999)
- Hayashi's limit realizability
- Bounded functional interpretation (Ferreira/Oliva 2004)

Proof interpretations as tool for generalizing proofs

$$
\begin{array}{ccc}
P & \xrightarrow{\;I\;} & P^I \\
G \downarrow & & \downarrow I^G \\
P^G & \xrightarrow{\;G^I\;} & (P^I)^G = (P^G)^I
\end{array}
$$

- Generalization $(P^I)^G$ of P^I: **easy!**
- Generalization P^G of P: **difficult!**

Proof Mining in Analysis I: concrete spaces

- Context: **continuous functions** between constructively represented **Polish spaces**.
- Uniformity w.r.t. parameters from **compact** Polish spaces.
- Extraction of **bounds** from **ineffective** existence proofs.

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
$BA :=$ basic arithmetic, HBC Heine/Borel compactness (SEQ$^-$ restricted sequential compactness) .
From a proof

$$BA + \text{ HBC(+SEQ}^-\text{)} \vdash \forall x \in P \forall y \in K \exists m \in \mathbb{N} A_\exists(x, y, m)$$

one can extract a closed term Φ of BA (+iteration)

$$BA(+\text{ IA }) \vdash \forall x \in P \forall y \in K \exists m \leq \Phi(f_x) A_\exists(x, y, m).$$

Important:

$\Phi(f_x)$ does not depend on $y \in K$ but on a representation f_x of x!

Logical comments

- Heine-Borel compactness = WKL (binary König's lemma).
  WKL $\vdash$ strict-$\Sigma^1_1 \leftrightarrow \Pi^0_1$
  (see applications in algebra by Coquand, Lombardi, Roy ...)
- Restricted sequential compactness = restricted arithmetical comprehension.

Limits of Metatheorem for concrete spaces

Compactness means constructively: **completeness** and **total boundedness**.

Necessity of completeness: The set $[0,2]_{\mathbb{Q}}$ is totally bounded and constructively representable and

$$\mathsf{BA} \vdash \forall q \in [0,2]_{\mathbb{Q}}\ \exists n \in \mathbb{N}(|q-\sqrt{2}| >_{\mathbb{R}} 2^{-n}).$$

However: no uniform bound on $\exists n \in \mathbb{N}$!

Necessity of total boundedness: Let B be the unit ball $C[0,1]$. B is bounded and constructively representable.
By Weierstraß' theorem

$$\mathrm{BA} \vdash \forall f \in B \exists n \in \mathbb{N}(n \text{ code of } p \in \mathbb{Q}[X] \text{ s.t. } \|p - f\|_\infty < \frac{1}{2})$$

but no uniform bound on $\exists n$: take $f_n := \sin(nx)$.

Necessity of $A_\exists$ '$\exists$-formula':

Let (f_n) be the usual sequence of spike-functions in $C[0,1]$, s.t. (f_n) converges pointwise but not uniformly towards 0. Then

$$\mathsf{BA} \vdash \forall x \in [0,1] \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} (|f_{n+m}(x)| \leq 2^{-k}),$$

but no uniform bound on '$\exists n$' (proof based on Σ^0_1-LEM).

Classically: uniform bound only if $(f_n(x))$ monotone (Dini): '$\forall m \in \mathbb{N}$' superfluous!

Necessity of $\Phi(f_x)$ depending on a representative of x :

Consider

$$\text{BA} \vdash \forall x \in \mathbb{R} \exists n \in \mathbb{N}((n)_{\mathbb{R}} >_{\mathbb{R}} x).$$

Suppose there would exist an $=_{\mathbb{R}}$-extensional computable $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ producing such a n. Then Φ would represent a **continuous** and hence **constant** function $\mathbb{R} \to \mathbb{N}$ which gives a contradiction.

MFI as numerical implication (K./Oliva,Proc.Steklov Inst.Math 2003)

X, K Polish spaces, K compact, $f : (X\times)K(\times\mathbb{N}) \to \mathbb{R}(X)$ (all BA-definable.

1) MFI transforms uniqueness statements

$$\forall x \in X, y_1, y_2 \in K\Big(\bigwedge_{i=1}^{2} f(x, y_i) =_{\mathbb{R}} 0 \to d_K(y_1, y_2) =_{\mathbb{R}} 0\Big)$$

into moduli of uniqueness $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall x \in X, y_1, y_2 \in K, \varepsilon > 0\Big(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon\Big).$$

More than 100-200 papers in the literature under the heading of strong uniqueness.

Let $\widehat{y} \in K$ be the unique root of $f(x,\cdot)$, y_ε an approximate root $|f(x,y_\varepsilon)| < \varepsilon$. Then $d_K(\widehat{y}, y_{\Phi(x,\varepsilon)}) < \varepsilon)$.

THEOREM 2 (K.,93)

For $\mathcal{T} = BA + \mathrm{HBC}(+\mathrm{SEC}^-)$ as before

$$\left\{ \begin{array}{l} \mathcal{T} \vdash \forall x \in X \exists! y \in K (F(x,y) =_{\mathbb{R}} 0) \\ \\ \exists\ BA(+\text{iter.})\text{-definable computable function } G : X \to K \text{ s.t.} \\ \\ BA(+\mathrm{IA}) \vdash \forall x \in X (F(x, G(x)) =_{\mathbb{R}} 0) \end{array} \right.$$

(X, K are BA-definable Polish spaces, K compact,
$F : X \times K \to \mathbb{R}$ BA-definable function).

2) M.f.i. transforms statements $f : K \to K$ is contractive

$$\forall x, y \in K(x \neq y \to d(f(x), f(y)) < d(x, y))$$

into moduli of contractivity $\alpha : \mathbb{R}^*_+ \to (0,1)$ (Rakotch)

$$\forall x, y \in K, \varepsilon > 0(d(x,y) > \varepsilon \to d(f(x), f(y)) < \alpha(\varepsilon) d(x,y)).$$

3) $f : K \times \mathbb{N} \to \mathbb{R}^+$ s.t. $(f(x,n))_{n\in\mathbb{N}}$ is non-increasing for $x \in K$. MFI transforms the statement

$$f(x,n) \stackrel{n\to\infty}{\to} 0$$

into a modulus of uniform convergence $\delta : \mathbb{Q}^*_+ \to \mathbb{N}$

$$\forall x \in K \forall \varepsilon > 0 \forall n \geq \delta(\varepsilon)(f(x,n) < \varepsilon).$$

(Numerous papers on such δ e.g. in metric fixed point theory).

The semi-classical case

Consider the **itjuitionistic** version $\mathbf{BA}_i$ of $\mathbf{BA}$.

AC $=$ full axiom of choice in all types

$\mathsf{CA}_\neg : \exists \Phi \forall x^\rho (\Phi(x) =_0 0 \leftrightarrow \neg A(x))$ A and ρ **arbitrary**.

Observation: $\mathsf{CA}_\neg$ implies WKL (and even UWKL) and the law of exluded middle for negated (and for $\exists$-free) formulas.

K., 1998: P Polish space, K a compact P-space, A **arbitrary**.
From a proof

$$\mathsf{BA}_i + \mathsf{AC} + \mathsf{CA}_{\neg} \vdash \forall x \in P \forall y \in K \exists m \in \mathbb{N} A(x, y, m)$$

one can extract a closed term Φ of BA_i

$$\mathsf{BA}_i + \mathsf{AC} + \mathsf{CA}_{\neg} \vdash \forall x \in P \forall y \in K \exists m \leq \Phi(f_x) A(x, y, m).$$

The purely intuitionistic case (without $\mathsf{CA}_{\neg}$) is known as **fan rule** (Troelstra 1977).

Proof Mining:
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Case study: strong unicity in L_1-approximation

P_n space of polynomials of degree $\leq n$, $f \in C[0,1]$,

$\|f\|_1 := \int_0^1 |f|, \quad dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1.$

Best approximation in the mean of $f \in C[0,1]$:

$$\forall f \in C[0,1] \exists! p_b \in P_n (\|f - p_b\|_1 = dist_1(f, P_n))$$

(existence **and** uniqueness: WKL!)

THEOREM 3 (K./Paulo Oliva, APAL 2003) Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\Psi(\omega, n, \varepsilon) := \min\{\frac{c_n \varepsilon}{8(n+1)^2}, \frac{c_n \varepsilon}{2}\omega_n(\frac{c_n \varepsilon}{2})\}, \textit{ where}$$

$$c_n := \frac{\lfloor n/2 \rfloor! \lceil n/2 \rceil!}{2^{4n+3}(n+1)^{3n+1}} \textit{ and}$$

$$\omega_n(\varepsilon) := \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}.$$

Then $\forall n \in \mathbb{N}, p_1, p_2 \in P_n$

$$\forall \varepsilon \in \mathbb{Q}^*_+ (\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) \leq \Psi(\omega, n, \varepsilon)) \rightarrow \|p_1 - p_2\|_1 \leq \varepsilon).$$

Comments on the result in the L_1-case

- Ψ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Ψ is **optimal**.
- Ψ allows the **first complexity upper bound** for the sequence of best L_1-approximations (p_n) in P_n of poly-time functions $f \in C[0,1]$:

THEOREM 4 (P. Oliva, MLQ 2003)
$(p_n)_{n\in\mathbb{N}}$ is strongly **NP** computable in $\mathbf{NP}[\mathrm{B}_f]$, where B_f is an oracle for a general left cut of $\|f - p\|_1$.

The nonseparable/noncompact case

Proposition 5 *Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| =$dist(x, C).*

Hence: if X is separable and complete and provably strictly convex and C compact, then one can extract a modulus of uniqueness.

Observation: compactness only used to exract uniform bound on strict convexity ($=$ **modulus of uniform convexity**) from proof of strict convexity.

Assume that X is uniformly convex with modulus η.

Then for $d \geq$dist(x, C) we have the following modulus of uniqueness (K.1990):

$$\Phi(\varepsilon) := \min\left(\frac{\varepsilon}{4}, d \cdot \frac{\eta(\varepsilon/(d+1))}{1-\eta(\varepsilon/(d+1))}\right).$$

Conclusion: neither compactness nor separability required!

Proposition 6 (Edelstein 1962) K compact metric space, $f : K \to K$ contractive, $x_n := f^n(x)$. Then for all $x \in K : x_n \to c$, where $c \in K$ is unique s.t. $f(c) = c$.

'Contractivity' (CT), 'uniqueness' (UN) and 'asymptotic regularity'

$$\text{(AS)} : d(x_n, f(x_n)) \to 0$$

have the logical form of the meta-theorem, whereas '(x_n) converges' has not. M.f.i.

- enriches f with a modulus of contractivity α,
- produces moduli Φ, δ of uniqueness and asymptotic regularity,
- builds modulus of convergence κ towards c out of Φ, δ:

$$\kappa(\varepsilon, D_K) = \frac{\log((1-\alpha(\varepsilon))\frac{\varepsilon}{2}) - \log D_K}{\log \alpha((1-\alpha(\varepsilon))\frac{\varepsilon}{2})} + 1.$$

Observation: If f is given with α only boundedness of K needed!

Remark 7

- Using a direct constructive proof one gets an improved modulus (Gerhardy/K., APAL 2006)

$$\delta(\alpha, b, \varepsilon) = \left\lceil \frac{\log \varepsilon - \log d_K}{\log \alpha(\varepsilon)} \right\rceil.$$

- Recently, E. Briseid obtained a quantitative version of a much more general fixed point theory due to Kincses and Totik for generalized p-contractive mappings (see his talk).
- P. Gerhardy (JMAA 2006) obtained an effective version of another generalization to Kirk's asymptotically contractive mappings. Some further results in this direction are due to E. Briseid.

In recent years (2000-2004) an extended case study in metric fixed point theory has been carried out (partly with P. Gerhardy, B. Lambov, L. Leuştean):

$(X, \|\cdot\|)$ normed linear space, $C \subset X$ convex, bounded,
$f : C \to C$ nonexpansive (n.e.)

$$\forall x, y \in C (\|f(x) - f(y)\| \le \|x - y\|).$$

More than 1000 papers on the fixed point theory of such mapping!

Our results concern the asymptotics

$$\|x_n - f(x_n)\| \to 0$$

of Krasnoselski-Mann iterations

$$x_0 := x,\ \ x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n), \lambda_n \in [0,1]$$

under various conditions on $(\lambda_n), (X, \|\cdot\|)$:

- (λ_k) is divergent in sum,
- $\forall k \geq k_0 (\lambda_k \leq 1 - \frac{1}{K})$ for some $K \in \mathbb{N}$.

THEOREM 8 (Borwein-Reich-Shafrir,1992)

For the Krasnoselski-Mann iteration (x_n) starting from $x \in C$ one has

$$\|x_n - f(x_n)\| \stackrel{n\to\infty}{\to} r_C(f),$$

where $r_C(f) := \inf_{x\in C} \|x - f(x)\|$.

COROLLARY 9 (Ishikawa,1976)

If $d(C) := diam(C) < \infty$, then $\|x_n - f(x_n)\| \stackrel{n\to\infty}{\to} 0$.

Proofs based on $(\|x_n - f(x_n)\|)$ being **non-increasing!**

- Also for hyperbolic spaces and directionally n.e. functions.
- For uniformly convex spaces: even asymptotically (quasi-)nonexpansive mappings.

Case studies II: general observations

1) Extraction works for general classes of (not necessarily Polish or constructive) spaces.

2) Uniformity even for metrically bounded (non-compact) spaces.

3) For bounded subsets C, assumptions

$$(1)\ \exists x \in C(f(x) =_{\mathbb{R}} 0)$$

can be reduced to their 'ε-weakenings'

$$(2)\ \forall \varepsilon > 0 \exists x \in C(|f(x)| < \varepsilon)$$

even when (1) is false while (2) is true!

Question: Are there logical meta-theorems to explain 1)-3)?

Hyperbolic Spaces

Definition 10 (Takahashi,Kirk,Reich)

A **hyperbolic space** is a triple (X, d, W) where (X, d) is metric space and $W : X \times X \times [0, 1] \to X$ s.t.

(i) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$,

(ii) $d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y)$,

(iii) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,

(iv) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$.

Examples: Open unit disk $D \subset \mathbb{C}$ and Hilbert ball with hyperbolic metric, Hadamard manifolds.

- a **CAT(0)-spaces (Gromov)** is a hyperbolic space (X, d, W) which satisfies the **CN**-inequality of Bruhat-Tits

$$\begin{cases} d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2) \to \\ d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \end{cases}$$

- **convex subsets of normed spaces** $=$ hyperbolic spaces (X, d, W) with homothetic distance (Machado (1973).

Notation: $(1-\lambda)x \oplus \lambda y := W(x, y, \lambda)$.

Functionals of finite type over $\mathbb{N}, X$

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map objects of type ρ to objects of type τ.

$\mathrm{PA}^{\omega,X}$ is the extension of Peano Arithmetic to all types.

Real numbers x are represented as **Cauchy sequences** (r_n) of rational numbers with rate of convergence 2^{-n} (can be encoded as functions $f^{\mathbb{N}\to\mathbb{N}}$ of type $\mathbb{N} \to \mathbb{N}$).

On these representatives one can define an equivalence relation

$$f =_{\mathbb{R}} g :\equiv \forall n^{\mathbb{N}} (|f(n+1) -_{\mathbb{Q}} g(n+1)| \leq 2^{-n}),$$

which expresses that f and g represent the same real number.

Classical Analysis

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$**+AC**$^{\mathbb{N}}$, where

AC$^{\mathbb{N}}$**: countable axiom of choice for all types**

which implies full comprehension for numbers:

$$\text{CA}: \ \exists f^{\mathbb{N}\to\mathbb{N}} \forall n^{\mathbb{N}} (f(n) = 0 \leftrightarrow A(n)), \ A \text{ arbitrary.}$$

Based on a quantifier-free extensionality rule

$$\frac{A_{qf} \to s =_\rho t}{A_{qf} \to r[s] =_\tau r[t]},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$x^X =_X y^X :\equiv d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}},$$

$$s =_{\rho\to\tau} t :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

The theory $\mathcal{A}^\omega[X,d,W]$ results by adding constants b_X, d_X, W_X axiom expressing that (X,d,W) is a nonempty b-bounded hyperbolic space.

Definition 11

$F \equiv \forall \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_{qf}(\underline{a})$) is a $\forall$-formula ($\exists$-formula) if F_{qf} is quantifier-free and $\underline{\sigma}$ are of the kind $\mathbb{N}, \mathbb{N} \to \mathbb{N}, X, \mathbb{N} \to X, X \to X$.

Definition 12 For $x \in [0, \infty) \subset \mathbb{R}$ define $(x)_\circ \in \mathbb{N}^\mathbb{N}$ by

$$(x)_\circ(n) := j(2k_0, 2^{n+1} - 1),$$

where

$$k_0 := \max k \Big[\frac{k}{2^{n+1}} \leq x \Big].$$

Lemma 13 1) If $x \in [0, \infty)$, then $(x)_\circ$ is a representative of x in the sense of our representation above.

2) If $x, x^* \in [0, \infty)$ and $x^* \geq x$ (in the sense of $\mathbb{R}$), then $(x^*)_\circ \geq_\mathbb{R} (x)_\circ$ and also $(x^*)_\circ \geq_1 (x)_\circ$.

3) $x \in [0, \infty]$, then $(x)_\circ$ is monotone, i.e. $\forall n \in \mathbb{N}((x)_\circ(n) \leq_0 (x)_\circ(n+1))$.

4) If $b \in \mathbb{N}$ $x, x^* \in [0, b]$ with $x^* \geq x$ then $(x^*)_\circ$ s-maj$_1$ $(x)_\circ$.

Definition 14 Let X be a non-empty set. The full set-theoretic type structure $\mathcal{S}^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$ over $\mathbb{N}$ and X is defined by

$$S_0 := \mathbb{N}, \quad S_X := X, \quad S_{\tau \to \rho} := S_\rho^{S_\tau}.$$

Here $S_\rho^{S_\tau}$ is the set of all set-theoretic functions $S_\tau \to S_\rho$.

Definition 15 A sentence of $\mathcal{L}(\mathcal{A}^\omega[X,d,W])$ holds in a bounded hyperbolic space (X,d,W) if it holds in the model of $\mathcal{A}^\omega[X,d,W]$ obtained by letting the variables range over the appropriate universes of $\mathcal{S}^{\omega,X}$ with the set X as the universe for the base type X where b_X is interpreted as some integer upper bound for d,

$$[W_X]_{\mathcal{S}^{\omega,X}}(x,y,\lambda^1) := W(x,y,r_{\tilde{\lambda}})$$
$$[d_X]_{\mathcal{S}^{\omega,X}}(x,y) := (d(x,y))_\circ,$$

where $r_{\tilde{\lambda}}$ is the real number $\in [0,1]$ represented by

$$\tilde{\lambda}^1 := \min_{\mathbb{R}}(1_{\mathbb{R}}, \max_{\mathbb{R}}(0_{\mathbb{R}}, \lambda)).$$

THEOREM 16 (K.,Trans.AMS,2005) Let P (resp.K) be a Polish (resp. compact) space and let $\underline{\tau}$ be of degree $(\mathbb{N}, X)$, $B_\forall$ ($C_\exists$) be a $\forall$-formula ($\exists$-formula). If

$$\forall x \in P \forall y \in K \forall \underline{z}^{\underline{\tau}} (\forall u^{\mathbb{N}} B_\forall(x, y, \underline{z}, u) \to \exists v^{\mathbb{N}} C_\exists(x, y, \underline{z}, v))$$

is provable in $\mathcal{A}^\omega[X, d, W]$, then there exists a computable $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all representatives $f_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $b \in \mathbb{N}$

$$\forall y \in K \forall \underline{z}^{\underline{\tau}} [\forall u \leq \Phi(f_x, b)\, B_\forall \to \exists v \leq \Phi(f_x, b)\, C_\exists]$$

holds in any (nonempty) b-bounded hyperbolic space (X, d, W).

Comments

- Also holds for **bounded convex subsets** of **normed spaces**.
- Applies to **uniformly convex** spaces: then bound depends on **modulus of convexity**.
- One can also treat **inner product spaces**.
- Applies to **totally bounded** spaces: then bound depends on **modulus of total boundedness**.
- Applies to **metric completion** of spaces.
- **Several spaces** and their **products** possible (with P. Gerhardy).

COROLLARY 17 (K.,Trans.AMS,2005)
If $\mathcal{A}^\omega[X,d,W]$ proves

$$\forall x \in P \forall y \in K \forall z^X, f^{X\to X}(f\ n.e. \wedge Fix(f) \neq \emptyset \to \exists v^{\mathbb{N}} C_\exists)$$

then there is a computable functional $\Phi(f_x, b)$ s.t. for all $x \in P$, f_x representative of x, $b \in \mathbb{N}$

$$\forall y \in K \forall z \in X \forall f : X \to X(f\ n.e. \to \exists v \leq \Phi(f_x, b)\, C_\exists)$$

holds in any b-bounded hyperbolic space (X, d, W).

Next Lecture: Much refined metatheorems for unbounded spaces (with P. Gerhardy)!

A uniform boundedness principle for X-type

THEOREM 18 (K.2006) The previous results also hold if the following (classically false) principle of **uniform $\exists$-uniform boundedness**

$$\exists\text{-UB}^X :\equiv \begin{cases} \forall y^{0\to 1}\Big(\forall k^0\, \forall x \leq_1 yk\, \forall \underline{z}^{\underline{\tau}}\, \exists n^0 A_\exists \to \\ \qquad \exists \chi^1\, \forall k^0\, \forall x \leq_1 yk\, \forall \underline{z}^{\underline{\tau}}\, \exists n \leq \chi(k)\, A_\exists\Big) \end{cases}$$

is added to $\mathcal{A}^\omega[X, d, W]$.

Here $\underline{\tau}$ are types of degree $(\mathbb{N}, X)$ and $A_\exists$ is an $\exists$-formula (extends results from K. 1996 for the case without X).

Limit of Metatheorems for case of abstract spaces

Full extensionality together with **Markov's principle** are in conflict with metatheorem:

$$\forall f^{X \to X}, x^X, y^X (x =_X y \to f(x) =_X f(y))$$

yields with Markov's principle

$$\forall f^{X \to X}, x^X, y^X, k \exists n (d_X(x,y) < 2^{-n} \to d_X(f(x), f(y)) < 2^{-k})$$

and hence with metatheorem:

All functions $f : X \to X$ have a common continuity modulus

(which only depends on the bound b of the metric).

Proof Mining: Applications of Proof Theory to Analysis III

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Refinements (Gerhardy/K.2005)

1) For $\rho \in T^X$ we define $\widehat{\rho} \in T$ by:

$$\widehat{0} := 0,\ \widehat{X} := 0,\ \widehat{\rho \to \tau} := \widehat{\rho} \to \widehat{\tau}.$$

2) $\gtrsim^a_\rho$ is the following ternary relation between functionals x, y of types $\widehat{\rho}, \rho$ and a^X of type X:

$$x^0 \gtrsim^a_0 y^0 :\equiv x \geq y$$

$$x^0 \gtrsim^a_X y^X :\equiv (x)_{\mathbb{R}} \geq_{\mathbb{R}} d_X(y, a)$$

$$x \gtrsim^a_{\rho\to\tau} y :\equiv \begin{cases} \forall z', z(z' \gtrsim^a_\rho z \to xz' \gtrsim^a_\tau yz) \land \\ \forall z', z(z' \gtrsim^a_{\widehat{\rho}} z \to xz' \gtrsim^a_{\widehat{\tau}} xz). \end{cases}$$

THEOREM 19 (Gerhardy/K.2005) Let $P, K, \tau, B_\forall, C_\exists$ be as before. If $\mathcal{A}^\omega[X, d, W]_{-b}$ proves

$$\forall x \in P \forall y \in K \forall z^\tau (\forall u^{\mathbb{N}} B_\forall(x, y, z, u) \to \exists v^{\mathbb{N}} C_\exists(x, y, z, v)),$$

then there exists a computable
$\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty hyperbolic space: for all representatives $f_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $z \in S_\tau, z^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X (z^* \gtrsim^a_\tau z)$:

$$\forall y \in K [\forall u \leq \Phi(f_x, z^*)\, B_\forall \to \exists v \leq \Phi(f_x, z^*)\, C_\exists].$$

Refined version of corollary 17 (Gerhardy/K.2005)

COROLLARY 20 1) Let $P, K, \tau, B_\forall, C_\exists$ be as before.If $\mathcal{A}^\omega[X,d,W]_{-b}$ proves

$$\forall x \in P \forall y \in K \forall z^X \forall f^{X\to X}(f \text{ n.e.} \wedge \forall u^0 B_\forall \to \exists v^0 C_\exists),$$

then there exists a computable functional $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t. for all representatives $r_x \in \mathbb{N}^\mathbb{N}$ of $x \in P$ and all $b \in \mathbb{N}$

$$\forall y \in K \forall z^X \forall f^{X\to X}(f \text{ n.e.} \wedge d_X(z, f(z)) \leq b$$
$$\wedge \forall u^0 \leq \Phi(r_x, b) B_\forall \to \exists v^0 \leq \Phi(r_x, b) C_\exists)$$

holds in all nonempty hyperbolic spaces (X, d, W).

Analogously, for $\mathcal{A}^\omega[X, d, W, \text{CAT}(0)]_{-b}$.

2) If additional parameter $\forall z'^X$ then $d_X(z,z') \leq_{\mathbb{R}} b$ needed.

3) If additional parameter $\forall c^{0\to X}$ then $\forall n(d_X(z,c(n)) \leq g(n))$ needed and bound depends on g.

4) 1., 2., 3. also hold if 'f n.e.' replaced by 'f Lipschitzian', 'f Hölder-Lipschitzian' or 'f uniformly continuous'. The the bound depends also on the constants/moduli.

5) 1., 2., 3. also hold if 'f n.e.' replaced by 'f weakly quasi-nonexpansive (with fixed point p)'. Then premise '$d_X(z,p) \leq b$' in the conclusion.

6) 1., 2., 3. also hold if 'f n.e.' is replaced by

$$\forall n^0, \tilde{z}^X (d_X(z,\tilde{z}) < n \to d_X(z,f(\tilde{z})) \leq \Omega(n)).$$

The bound only depends on Ω instead of b.

Elimination of fixed points (Gerhardy/K.2005)

Definition 21 $\mathcal{H}$ = formulas with prenexation $\exists x_1^{\rho_1} \forall y_1^{\tau_1} \ldots \exists x_n^{\rho_n} \forall y_n^{\tau_n} F_\exists(\underline{x}, \underline{y})$, where $F_\exists$ is an $\exists$-formula, ρ_i of degree 0 and τ_i of degree 1 or $(0, X)$.

COROLLARY 22 Let A be in $\mathcal{H}$. If $\mathcal{A}^\omega[X, d, W]_{-b}$ proves

$$\forall x \in P \, \forall y \in K \, \forall z^X, f^{X \to X} (f \text{ n.e.} \wedge Fix(f) \neq \emptyset \to A)$$

then the following holds in every hyperbolic space:

$$\forall x \in P \, \forall y \in K \, \forall z^X, f^{X \to X}$$
$$(f \text{ n.e.} \wedge \exists b^0 \forall \varepsilon > 0 (Fix_\varepsilon(f, z, b) \neq \emptyset) \to A).$$

Analogously, for $\mathcal{A}^\omega[X, d, W, \mathrm{CAT}(0)]_{-b}$.
Similarly for Lipschitzian etc. functions.

Comments

- Also holds for **normed spaces** (additional condition '$\|x\| \leq b$'), **convex subsets** $C \subset X$, **CAT(0)-spaces**.
- One can also treat **inner product spaces**.
- Applies to **uniformly convex** spaces: then bound depends on **modulus of convexity**.
- Applies to **totally bounded** spaces: then bound depends on **modulus of total boundedness**.
- **Several spaces** and their **products** possible.
- Applies to **metric completion** of spaces.

General assumptions

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.
- (λ_n) is a sequence in $[0, 1]$ that is **bounded away from** 1 and **divergent in sum**.
- $x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_n f(x_n)$ (**Krasnoselski-Mann iter.**).

THEOREM 23 (Ishikawa 1976, Goebel/Kirk 1983)

If (x_n) is bounded, then $d(x_n, f(x_n)) \to 0$.

THEOREM 24 (Borwein/Reich/Shafrir 1992)

$$d(x_n, f(x_n)) \to r(f) := \inf_{y \in X} d(y, f(y)).$$

Corollary to refined metatheorem

COROLLARY 25 (Gerhardy/K.2005)

If $\mathcal{A}^{\omega}[X,d,W]_{-b}$ proves

$$\forall x \in P \forall y \in K \forall z^{X}, f^{X\to X}(f\ n.e \to \exists v^{\mathbb{N}} C_{\exists})$$

then there is a **computable functional** $\Phi(g_x, b)$ s.t. for all $x \in P, g_x$ representative of x, $b \in \mathbb{N}$

$$\forall y \in K \forall z \in X \forall f^{X\to X}(f\ n.e. \wedge d(z,f(z)) \le b \to \exists v \le \Phi(g_x,b)\, C_{\exists})$$

holds in **any** nonempty hyperbolic space (X,d,W).

Application 1

Let $(\lambda_n)_{n\in\mathbb{N}} \subset [0, 1-\frac{1}{k}]$ with $\forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\alpha(n)} \lambda_i)$ and

(x_n) the Krasnoselski-Mann iteration of f starting from x.

Then by Ishikawa(76),Goebel/Kirk(83), $\mathcal{A}^\omega[X,d,W]_{-b}$ proves

$$(x_n) \text{ bounded} \wedge f \text{ n.e.} \rightarrow \lim_{n\to\infty} d(x_n, f(x_n)) = 0.$$

By the cor. there is a computable Φ s.t. $\forall l \forall m \geq \Phi(k,\alpha,b,l)$

$$(x_n)\ b\text{-bounded} \wedge f \text{ n.e.} \rightarrow d(x_m, f(x_m)) < 2^{-l}$$

holds in any (nonempty) hyperbolic space (X,d,W).

(normed case: K., Numer.Funct.Opt.2001/JMAA 2003,

hyperbolic, directionally n.e.: Leustean/K., Abstr.Appl.Anal.2003)

Known uniformi.ty results in the bounded case

blue = hyperbolic, green = dir.nonex., red = both.

- Krasnoselski(1955):Xunif.convex,C compact,$\lambda_k=\frac{1}{2}$,no uniform.
- Browder/Petryshyn(1967):Xunif.convex,$\lambda_k = \lambda$, no uniformity.
- Groetsch(1972): X unif. convex, allg. λ_k, X, no uniformity
- Ishikawa (1976): No uniformity
- Edelstein/O'Brien (1978): Uniformity w.r.t. $x_0 \in C$ $(\lambda_k := \lambda)$
- Goebel/Kirk (1982): Uniformity w.r.t. x_0 **and** f. General λ_k
- Kirk/Martinez (1990): Uniformity for unif. convex $X, \lambda := 1/2$
- Goebel/Kirk (1990): Conjecture: no uniformity w.r.t. C
- Baillon/Bruck (1996): Uniformity w.r.t. x_0, f, C for $\lambda_k := \lambda$
- Kirk (2001): Uniformity w.r.t. x_0, f for constant λ
- Kohlenbach (2001): Full uniformity for general λ_k
- K./Leustean (2003): Full uniformity for general λ_k

Application 2: The Borwein-Reich-Shafrir Theorem

THEOREM 26 (Borwein-Reich-Shafrir 1992)

(X, d, W) hyperbolic space, $f : X \to X$ n.e. For the Krasnoselski-Mann iteration (x_n) starting from $x \in X$ one has

$$d(x_n, f(x_n)) \overset{n\to\infty}{\to} r(f),$$

where $r(f) := \inf_{y \in X} d(y, f(y))$.

Since $(d(x_n, f(x_n)))$ is non-increasing, the BRS-Theorem formalizes as either

$$(a)\ \forall\varepsilon>0\exists n\in\mathbb{N}\forall x^*\in X(d(x_n,f(x_n))<d(x^*,f(x^*))+\varepsilon)$$

or

$$(b)\ \forall\varepsilon>0\forall x^*\in X\ \exists n\in\mathbb{N}(d(x_n,f(x_n))<d(x^*,f(x^*))+\varepsilon).$$

Only (b) meets the specification in the meta-theorem.

The refined metatheorem predicts a uniform bound depending on x,x^*,f only via $b\geq d(x,x^*),d(x,f(x))$ and on (λ_k) only via $k,\alpha:$

THEOREM 27 (K./Leustean,AAA2003)
Let (X, d, W) be a hyperbolic space, $(\lambda_n)_{n\in\mathbb{N}}, k, \alpha$ as before. $f : X \to X$ n.e., $x, x^* \in X$ with $d(x, x^*), d(x, f(x)) \le b$. Then

$$\forall \varepsilon > \forall n \ge \Psi(k, \alpha, b, \varepsilon)\ (d(x_n, f(x_n)) < d(x^*, f(x^*)) + \varepsilon),$$

where

$\Psi(k, \alpha, b, \varepsilon) := \widehat{\alpha}(\lceil 2b \cdot \exp(k(M + 1))\rceil \dot{-} 1, M),$
with $M := \left\lceil \frac{1+2b}{\varepsilon} \right\rceil$ and
$\widehat{\alpha}(0, M) := \tilde{\alpha}(0, M),\ \widehat{\alpha}(m + 1, M) := \tilde{\alpha}(\widehat{\alpha}(m, M), M)$ with
$\tilde{\alpha}(m, M) := m + \alpha(m, M)\ \ (m \in \mathbb{N}).$

Definition 28

$f : X \to X$ is directionally nonexpansive (Kirk 2000) if

$$\forall x \in X \forall y \in [x, f(x)](d(f(x), f(y)) \leq d(x, y)).$$

THEOREM 29 (K./Leustean,AAA2003)

The previous theorem (and bound) also holds for directionally nonexpansive mappings of $d(x, x^*) \leq b$ is strengthened to $d(x_n, x_n^*) \leq b$ for all n.

Applications of the uniform BRS: The approximate fixed point property for product spaces

Let (X, ρ, W) be a hyperbolic space and M a metric space with AFPP for nonexpansive mappings.
Let $\{C_u\}_{u \in M} \subseteq X$ be a family of convex sets such that there exists a nonexpansive *selection* function $\delta : M \to \bigcup_{u \in M} C_u$ with

$$\forall u \in M (\delta(u) \in C_u).$$

Consider subsets of $(X \times M)_\infty$:

$$H := \{(x, u) : u \in M, x \in C_u\}.$$

If $P_1 : H \to \bigcup_{u\in M} C_u, P_2 : H \to M$ are the projections, then for any nonexpansive function $T : H \to H$ w.r.t. d_∞ satisfying

$$(*)\ \forall (x,u) \in H\ ((P_1 \circ T)(x,u) \in C_u)$$

we can define for each $u \in M$, the nonexpansive function

$$T_u : C_u \to C_u, \quad T_u(x) = (P_1 \circ T)(x,u).$$

We denote the Krasnoselski-Mann iteration starting from $x \in C_u$ and associated with T_u by (x_n^u) ((λ_n) as before).

$r_S(F)$ always denotes the **minimal displacement** of F on S.

Application 3 (K./Leustean, NA 2006)

THEOREM 30 (K./Leustean) Assume that $T : H \to H$ is nonexpansive with $(*)$ and $\sup_{u \in M} r_{C_u}(T_u) < \infty$.
Suppose there exists $\varphi : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ s.t.

$$\forall \varepsilon > 0 \, \forall v \in M \, \exists x^* \in C_v \quad (\rho(\delta(v), x^*) \le \varphi(\varepsilon) \wedge \\ \wedge \, \rho(x^*, T_v(x^*)) \le \sup_{u \in M} r_{C_u}(T_u) + \varepsilon).$$

Then
$$r_H(T) \le \sup_{u \in M} r_{C_u}(T_u).$$

THEOREM 31 (K./Leustean) Assume that there is $b > 0$ s.t.

$$\forall u \in M \exists x \in C_u(\rho(\delta(u), x) \le b \wedge \forall n, m \in \mathbb{N}(\rho(x^u_n, x^u_m) \le b).$$

Then $r_H(T) = 0$.

COROLLARY 32 (K./Leustean) Assume that $\{C_u\}_{u\in M} \subseteq X$ is a family of **bounded** convex sets such that there is $b > 0$ with the property that

$$\forall u \in M(diam(C_u) \leq b).$$

Then H has AFPP for nonexpansive mappings $T : H \to H$ satisfying $(*)$.

COROLLARY 33 (Kirk 2004) If $C_u := C$ constant and C bounded, then H has the approximate fixed point property.

Uniform approximate fixed point property

Let $\mathcal{F}$ be a class of functions $T : C \to C$.

C has uniform approximate fixed point property (UAFPP) if for all $\varepsilon > 0$ and $b > 0$ there exists $M > 0$ s.t. for any point $x \in C$ and $T \in \mathcal{F}$,

$$\rho(x, T(x)) \leq b \Rightarrow \exists x^* \in C(\rho(x, x^*) \leq M \wedge \rho(x^*, T(x^*)) < \varepsilon).$$

C has the uniform asymptotic regularity property if for all $\varepsilon > 0$ and $b > 0$ there exists $N \in \mathbb{N}$ s.t. for any point $x \in C$ and $T : C \to C$,

$$\rho(x, T(x)) \leq b \Rightarrow \forall n \geq N(\rho(x_n, T(x_n)) < \varepsilon),$$

where (x_n) is the Krasnoselski iteration ($\lambda_n = \frac{1}{2}$).

THEOREM 34 (K./Leustean) The following are equivalent:

1) C has UAFPP for nonexpansive functions;

2) C has the uniform asymptotic regularity property ;

One can prove that the following naive version of UAFPP

$$\forall \varepsilon > 0 \exists M > 0 \forall x \in C \forall T : C \to C \text{ nonexpansive}$$
$$\exists x^* \in C(\rho(x, x^*) \leq M \wedge \rho(x^*, T(x^*)) < \varepsilon).$$

even just for constant functions T is equivalent to C being bounded.

C has uniform fixed point property (UFPP) for $\mathcal{F}$ if for all $b > 0$ there exists $M > 0$ s.t. for any point $x \in C$ and $T \in \mathcal{F}$,

$$\rho(x, T(x)) \leq b \Rightarrow \exists x^* \in C(\rho(x, x^*) \leq M \wedge T(x^*) = x^*).$$

Example:

Assume that (X, ρ) is a complete metric space. Let $\mathcal{F}$ be the class of contractions with a common contraction constant k. Then each closed subset C of X has the UFPP for $\mathcal{F}$.

Open problem: Are there unbounded convex sets X (in suitable hyperbolic spaces) with the UAFPP for nonexpansive functions?

Proof Mining: Applications of Proof Theory to Analysis IV

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Application 4

Let $(X, d, W), (\lambda_n), f : X \to X, (x_n)$ be as in the Ishikawa-Goebel-Kirk theorem.

THEOREM 35 (Ishikawa, Goebel, Kirk) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

Proof: By Ishikawa-Goebel-Kirk: $d(x_n, f(x_n)) \to 0$. (x_n) has subsequence (x_{n_k}) whose limits $\widehat{x}$ must be a fixed point. Since $d(x_{n+1}, \widehat{x}) \leq d(x_n, \widehat{x})$, already (x_n) converges to $\widehat{x}$. $\square$

Proposition 36 (K., NA2005) There exists a computable sequence $(f_l)_{l\in\mathbb{N}}$ of nonexpansive functions $f_l : [0,1] \to [0,1]$ such that for $\lambda_n := \frac{1}{2}$ and $x_0^l := 0$ and the corresponding Krasnoselski iterations (x_n^l) there is **no computable function** $\delta : \mathbb{N} \to \mathbb{N}$ such that

$$\forall m \geq \delta(l)(|x_m^l - x_{\delta(l)}^l| \leq \frac{1}{2}).$$

Problem:

Cauchy property $\forall\exists\forall$ rather than $\forall\exists$ (asymptotic regularity).

Best possible: Bound on the no-counterexample interpretation:

$$\text{(H)}\ \forall g : \mathbb{N} \to \mathbb{N} \forall k \exists n \forall j_1, j_2 \in [n; n + g(n)](d(x_{j_1}, x_{j_2}) < 2^{-k}).$$

THEOREM 37 (K.,NA2005) There exists a computable functional Ψ such that for any rate of asymptotic regularity Φ and any modulus of total boundedness α for C, any g, k :

$$\exists n \leq \Psi(\Phi, \alpha, g, k) \forall j_1, j_2 \in [n; n + g(n)](d(x_{j_1}, x_{j_2}) < 2^{-k}).$$

Ψ has any **uniformity** Φ has!

Asymptotic regularity special case where $g(n) := 1$ since $d(x_{n+1}, x_n) = \lambda_n d(x_n, f(x_n))$.

For $g(n) := C$ (constant): no total boundedness required.

For general g : total boundedness known to be necessary.

The bound in the previous theorem is given by

$$\Psi(\Phi, \alpha, g, k) := \max_{i \le \alpha(k+3)} \Psi_0(i, k, g, \Phi),$$

where

$$\begin{cases} \Psi_0(0, k, g, \Phi) := 0 \\ \Psi_0(n+1, k, g, \Phi) := \Phi\left(2^{-k-2}/(\max_{l \le n} g(\Psi_0(l, k, g, \Phi)) + 1)\right). \end{cases}$$

Application 5: Groetsch's theorem

THEOREM 38 (K.,JMAA 2003)

Let $(X, \|\cdot\|)$ be a uniformly convex normed linear space with modulus of uniform convexity η, $d > 0$, $C \subseteq X$ a (non-empty) convex subset, $f : C \to C$ nonexpansive and $(\lambda_k) \subset [0,1]$ and $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\Big(\sum_{s=0}^{\gamma(n)} \lambda_s(1-\lambda_s) \geq n\Big).$$

Then for all $x \in C$ such that

$$\forall \varepsilon > 0 \exists y \in C(\|x-y\| \leq d \wedge \|y - f(y)\| < \varepsilon)$$

one has

$$\forall \varepsilon > 0 \forall k \geq h(\varepsilon, d, \gamma)(\|x_k - f(x_k)\| \leq \varepsilon),$$

where $h(\varepsilon, d, \gamma) := \gamma\left(\frac{3(d+1)}{2\varepsilon \cdot \eta(\frac{\varepsilon}{d+1})}\right)$.
Moreover, if $\eta(\varepsilon)$ can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with

$$(*)\ \forall \varepsilon_1, \varepsilon_2 \in (0,2](\varepsilon_1 \geq \varepsilon_2 \rightarrow \tilde{\eta}(\varepsilon_1) \geq \tilde{\eta}(\varepsilon_2)),$$

then the bound $h(\varepsilon, d, \gamma)$ can be replaced by

$$\tilde{h}(\varepsilon, d, \gamma) := \gamma\left(\frac{d+1}{2\varepsilon \cdot \tilde{\eta}(\frac{\varepsilon}{d+1})}\right).$$

Recenctly: generalization to uniformly convex hyperbolic spaces and **quadratic bounds for CAT(0)-spaces** (L. Leustean 2005, see his talk).

Definition 39 (Goebel/Kirk,1972) $f : C \to C$ is said to be **asymptotically nonexpansive with sequence** $(k_n) \in [0,\infty)^{\mathbb{N}}$ if $\lim\limits_{n\to\infty} k_n = 0$ and

$$\|f^n(x) - f^n(y)\| \leq (1 + k_n)\|x - y\|, \ \forall n \in \mathbb{N}, \forall x, y \in C.$$

$$x_0 := x \in C, \ x_{n+1} := (1 - \lambda_n)x_n + \lambda_n f^n(x_n).$$

Let $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$ be such that

$$\forall q \in \mathbb{Q}^*_+ \exists m \leq \Phi(q) \ (\|x_m - f(x_m)\| \leq q).$$

THEOREM 40 (K.,NA2005) The previous theorem also holds for asymptotically nonexpansive mappings in normed spaces (with a more complicated $\Psi(\Phi, \alpha, k, g)$).

Asymptotically quasi-nonexpansive mappings

Definition 41 (Schu,1991) $f : C \to C$ is said to be **uniformly λ-Lipschitzian** $(\lambda > 0)$ if

$$\|f^n(x) - f^n(y)\| \leq \lambda \|x - y\|, \ \forall n \in \mathbb{N}, \forall x, y \in C.$$

Definition 42 (Dotson,1970) $f : C \to C$ is **quasi-nonexpansive** if

$$\|f(x) - p\| \leq \|x - p\|, \ \forall x \in C, \forall p \in Fix(f).$$

Definition 43 (Shrivastava,1982) $f : C \to C$ is **asymptotically quasi-nonexpansive** with $k_n \in [0, \infty)^{\mathbb{N}}$ if $\lim\limits_{n \to \infty} k_n = 0$ and

$$\|f^n(x) - p\| \leq (1 + k_n)\|x - p\|, \ \forall n \in \mathbb{N}, \forall x \in X, \forall p \in Fix(f).$$

For asymptotically quasi-nonexpansive mappings $f : C \to C$ the **Krasnoselski-Mann iteration with errors** is

$$x_0 := x \in C,\ x_{n+1} := \alpha_n x_n + \beta_n f^n(x_n) + \gamma_n u_n,$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $u_n \in C$.

Relying on previous results of Opial(67), Dotson(70), Schu(91), Rhoades(94), Tan/Xu(94), Xu(98),Zhou(01/02) we have

Definition 44

$f : C \to C$ *is* **asymptotically weakly quasi-nonexpansive** *if*

$$\exists p \in Fix(f) \wedge \forall x \in C \forall n \in \mathbb{N}(\|f^n(x) - p\| \le (1 + k_n)\|x - p\|).$$

THEOREM 45 (K./Lambov,2004) Let $(X, \|\cdot\|)$ be a uniformly convex space and $C \subseteq X$ convex. $(k_n) \subset \mathbb{R}_+$ with $\sum k_n < \infty$. Let $k \in \mathbb{N}$ and $\alpha_n, \beta_n, \gamma_n \in [0,1]$ such that $1/k \le \beta_n \le 1 - 1/k$, $\alpha_n + \beta_n + \gamma_n = 1$ and $\sum \gamma_n < \infty$. $f : C \to C$ uniformly Lipschitzian and asymptocially weakly quasi-nonexpansive and (u_n) be a bounded sequence in C. Then the following holds:

$$\|x_n - f(x_n)\| \to 0.$$

Application 6
(Proc. Fixed Point Theory, Yokohama Press 2004)

THEOREM 46 (K./Lambov) $(X, \|\cdot\|)$ uniformly convex with η. $C \subseteq X$ convex, $x \in C, f : C \to C, \alpha_n, \beta_n, \gamma_n, k_n, u_n$ as before with $\sum \gamma_n \leq E$, $\sum k_n \leq K, \forall n(\|u_n - x\| \leq u)$.
If f is λ-uniformly Lipschitzian and

$$\forall \varepsilon > 0 \exists p_\varepsilon \in C \begin{pmatrix} \|f(p_\varepsilon) - p_\varepsilon\| \leq \varepsilon \wedge \|p_\varepsilon - x\| \leq d \wedge \\ \forall y \in C \forall n(\|f^n(y) - f^n(p_\varepsilon)\| \leq (1 + k_n)\|y - p_\varepsilon\|) \end{pmatrix}.$$

Then

$$\forall \varepsilon > 0 \exists n \leq \Phi(\|x_n - f(x_n)\| \leq \varepsilon),$$

where

$$\Phi := \Phi(K, E, k, d, \lambda, \eta, \varepsilon) :=$$
$$\left\lceil \frac{3(5KD+6E(U+D)+D)k^2}{\tilde{\varepsilon}\eta(\tilde{\varepsilon}/(D(1+K)))} \right\rceil + 1,$$

$$D := e^K(d + EU), U := u + d,$$

$$\tilde{\varepsilon} := \varepsilon/(2(1 + \lambda(\lambda + 1)(\lambda + 2))).$$

Application 7 (B. Lambov, ENTCS2005)

THEOREM 47 (Hillam 1975) Let $f : [u, v] \to [u, v]$ be Lipschitz continuous with constant L. For any $x_0 \in [u, v]$ define

$$x_{n+1} := (1 - \lambda)x_n + \lambda f(x_n), \text{ where } \lambda := 1/(L + 1).$$

Then (x_n) converges to a fixed point of f.

Based on an extension of a result due to Matiyasevich, B. Lambov proved

THEOREM 48 Under the same assumptions as above. If f has a unique fixed point with modulus of uniqueness η then

$$\forall m > \Phi(k)(|x_m - x_{\Phi(k)}| \leq 2^{-k}),$$

where $\Phi(k) := 2(v - u)2^{\eta(k+\lceil \log_2(L+1) \rceil)}$.

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