Noetherian rings

Perdry – Schuster

(work in progress)

Castro Urdiales, MAP 2006

Noetherian rings (classical definition)

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A ring R is said to be Noetherian if the poset $(\mathcal{J}_\mathsf{R}, \subseteq)$ of all ideals of R satisfies ACC.

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The key result of the theory of Noetherian rings is the following theorem.

Noether's theorem If R is a Noetherian ring, then so is R[X].

From the constructive point of view...

...what does that mean?

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Even with this restriction, the rings $\mathbb{Z}$ or $\mathbb{Q}$ fail to be Noetherian.

It is worth remarking that the proof of Noether's Theroem is constructive; the point is that the only ring which verifies constructively the hypotheses is the trivial ring $\{0\}$.

We need a new definition for Noetherian.

The key criteria for a good new definition of Noetherian rings are the following:

- It must be, from the point of view of classical mathematics, equivalent to the classical definition.
- It must hold, from the constructive point of view, at least for fields and for most usual Noetherian rings.
- One must be able to prove constructively that if it holds for a ring R, it is inherited by $R[X]$.

Richman/Seidenberg

In 1974, Fred Richman and Abraham Seidenberg gave the following version of the ascending chain condition.

RS If $(a_i)_{i\in\mathbb{N}}$ is a weakly increasing sequence, there exists some index $n \in \mathbb{N}$ such that $a_n = a_{n+1}$.

From the classical viewpoint the two conditions ACC and RS are equivalent.

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From the classical viewpoint the two conditions ACC and RS are equivalent.

Definition Let R be a ring; the set of finitely generated ideals of R is denoted $\mathcal{J}_{\mathsf{R}}$. The ring R is said to be RS–Noetherian if the poset $(\mathcal{J}_{\mathsf{R}}, \subseteq)$ satisfies RS.

The key result If R is coherent and RS–Noetherian, so is R[X]. Moreover, if R is strongly discrete, so is R[X].

A ring R is coherent if for all $a_1, \ldots, a_n \in \mathsf{R}$, the kernel of the map

$$\begin{array}{ccc} \mathsf{R}^n & \longrightarrow & \mathsf{R} \\ (x_1, \ldots, x_n) & \mapsto & a_1 \cdot x_1 + \cdots + a_n \cdot x_n \end{array}$$

is finitely generated. This submodule of R^n is the syzygy module of the ideal $\langle a_1, \ldots, a_n \rangle \in \mathcal{I}_{\mathsf{R}}$.

The ring R is said to be strongly discrete if, given $a_1, \ldots, a_n$ and x in R, one can decide whether $x \in \langle a_1, \ldots, a_n \rangle$ or not.

Note that in classical math both of these statements hold for any Noetherian ring.

A property of Noetherian rings

Theorem Let I be an ideal in a Noetherian ring R. There exists finitely many prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_q$ containing I, s.t. if $\mathfrak{P}$ is a prime ideal containing I, there exists i s.t. $I \subseteq \mathfrak{P}_i \subseteq \mathfrak{P}$.

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Classical Algebra

Let $\mathfrak{F}$ be the family of all ideals not satisfying this property. R is Noetherian, so if $\mathfrak{F}$ is nonempty we can choose a maximal element I in $\mathfrak{F}$. I is in $\mathfrak{F}$, so it is not prime; take $a, b \in \mathsf{R}$ s.t. $ab \in \mathsf{I}$ and $a, b \notin \mathsf{I}$.

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The ideals $\mathsf{I} + a\mathsf{R}$ and $\mathsf{I} + b\mathsf{R}$ are strictly greater than I, hence not in $\mathfrak{F}$; there exists finitely many primes $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ and $\mathfrak{P}_{r+1}, \ldots, \mathfrak{P}_q$ containing each, with the property stated in the lemma.

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Any prime ideal $\mathfrak{P}$ above I contains $\mathsf{I} + a\mathsf{R}$ or $\mathsf{I} + b\mathsf{R}$, so contains one of the $\mathfrak{P}_1, \ldots, \mathfrak{P}_q$; this is a contradiction, so $\mathfrak{F}$ is empty.

Computer Algebra

We say that we have a strong primality test in R if we can decide whether a finitely generated ideal I of R is prime or not, and if not, to produce $a, b \in$ R s.t. $ab \in$ I and $a, b \notin$ I.

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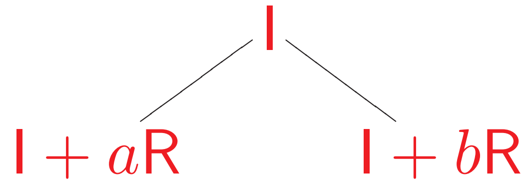
We say that we have a strong primality test in R if we can decide whether a finitely generated ideal I of R is prime or not, and if not, to produce $a, b \in \mathsf{R}$ s.t. $ab \in \mathsf{I}$ and $a, b \notin \mathsf{I}$.

Algorithm Let I be an ideal. If I is prime, let $\mathfrak{P}_1 = \mathsf{I}$ and we are done. If not, let $a, b \in \mathsf{R}$ s.t. $ab \in \mathsf{I}$ and $a, b \notin \mathsf{I}$. Begin to construct the following tree:



and apply the test to each leaf of the tree.

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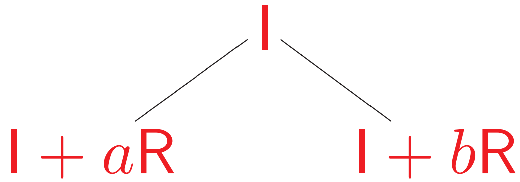
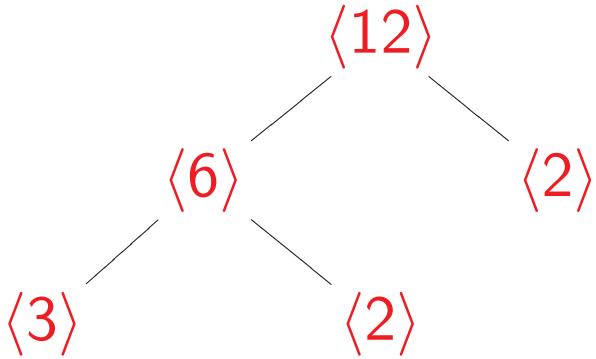


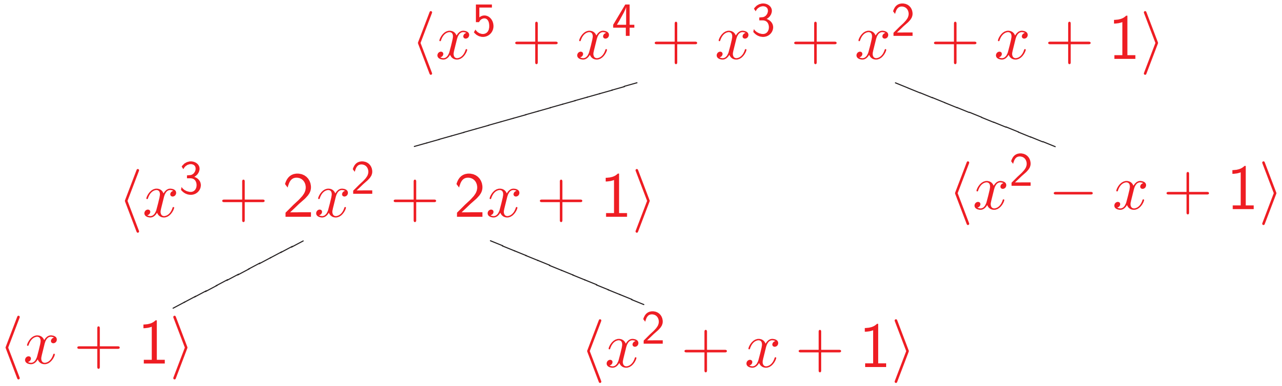
and apply the test to each leaf of the tree.

In this way, we construct a binary tree, with nodes labelled by ideals of R, such that, along each branch of it, there is an increasing sequence of ideals. Then each branch is finite; so the tree is finite. The ideals labelling the leaves of this tree are the minimal primes containing I.

Examples Ideals of $\mathbb{Z}$:



and ideals of $\mathbb{Q}[x]$:



Constructive Algebra

The Richman/Seidenberg theory of Noetherian rings allows to prove, for a wide class of rings, that the branches of our binary tree are finite.

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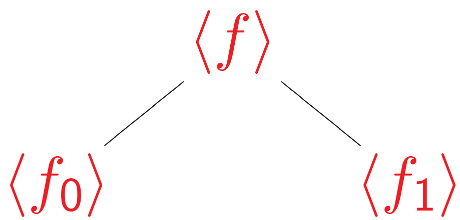
In the case $\mathsf{R} = \mathbb{Q}[x]$ this can be proved directly by induction on the degree of the polynomial generating the ideal.

If $\mathsf{I} = \langle f \rangle$, and $n = \deg f$, the tree starts like



with $\deg f_0 < n$ and $\deg f_1 < n$. By induction, the two subtrees starting by $\langle f_0 \rangle$ and $\langle f_1 \rangle$ are finite, and so is this tree.

The same proof can be done for ideals of $\mathbb{Z}$, replacing the degree by $\langle a \rangle \mapsto \mid a \mid$.

A possible solution: strongly Noetherian rings

Definition Let $(\mathsf{E}, \leq)$ be a poset. A subset H of E is hereditary if

$$\forall x, \left(\{y \,:\, y < x\} \subseteq \mathsf{H} \Longrightarrow x \in \mathsf{H} \right).$$

The poset E is well-founded if the only hereditary subset of E is $\mathsf{H} = \mathsf{E}$.

A totally ordered well-founded set is well-ordered.

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Definition Let $(\mathsf{E}, \leq)$ be a poset; the condition $\mathsf{STRONG}(\mathsf{E})$ holds if there exists (explicitly) an increasing map ϕ from $(\mathcal{I}_{\mathsf{R}}, \subseteq)$ to a well-ordered set $(\mathsf{E}, \leq)$.

Definition A strongly discrete and coherent ring R is strongly Noetherian if the poset $\mathsf{STRONG}(\mathcal{I}_{\mathsf{R}}, \supseteq)$ holds.

Remark If R is a strongly Noetherian ring, then the poset $(\mathcal{I}_{\mathsf{R}}, \supseteq)$ is well-founded.

Examples

- The ring $\mathbb{Z}$ is strongly noetherian: each finitely generated ideal is principal, so we map $\mathcal{I}_{\mathbb{Z}}$ to $\mathbb{N} \cup \{+\infty\}$, by $(0) \mapsto +\infty$ and for $a \neq 0$, $(a) \mapsto |\, a \,|$.
- Let F be a (discrete) field. The ring F[X] is strongly Noetherian; again, we map $\mathcal{I}_{\mathsf{F[X]}}$ to $\mathbb{N} \cup \{+\infty\}$, by $(0) \mapsto +\infty$ and for $f \neq 0$, $(f) \mapsto \deg f$.

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The key result If R is a coherent, strongly discrete and strongly Noetherian ring, so is $\mathsf{R[X]}$.

An other possible solution: a restricted fan condition

Let E be a poset. A finitely branching tree T with nodes labelled by elements of a poset E is said to be non-increasing (resp. decreasing) in E if the labelling $\phi : T \longrightarrow E$ is a non-increasing (resp. decreasing) map.

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We say that FAN(E) holds if, and only if, every non-increasing finitely branching tree T labelled with has a finite depth.

Note that in the particular case of a decreasing tree T in E, FAN(E) implies that all branches of the tree have length smaller than N.

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...the proofs of all these "key results" are very similar.

Is it possible to save some work here?

Acceptable properties

Let $(\mathsf{E}_i, \leq_i)_{i \in \mathsf{I}}$ be a family of posets, indexed by a poset $(\mathsf{I}, \leq)$. We denote by $\sum_{i \in \mathsf{I}} \mathsf{E}_i$ the disjoint union of the E_i's ordered by

$$x \in \mathsf{E}_i \preceq y \in \mathsf{E}_j \iff i < j \text{ or } i = j \wedge x \leq_i y\,.$$

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Let $\mathcal{P}$ be a property of posets (is E is a poset, $\mathcal{P}(\mathsf{E})$ may or may not hold constructively). It is an acceptable property if the following hold:

- $\mathcal{P}(\mathsf{E}) \Longrightarrow \textsc{rs}(\mathsf{E})$.
- If there is an increasing map from E to F and $\mathcal{P}(\mathsf{F})$ holds, then $\mathcal{P}(\mathsf{E})$ holds.
- If $(\mathsf{E}_i, \leq_i)_{i \in \mathsf{I}}$ if family of posets, such that $\mathcal{P}(\mathsf{I})$ holds and for all i, $\mathcal{P}(\mathsf{E}_i)$ holds. Then $\mathcal{P}(\sum_{i \in \mathsf{I}} \mathsf{E}_i)$ holds.
- $\mathcal{P}(\mathbb{N})$ holds constructively.

The key of all key results

Let $\mathcal{P}$ be an acceptable property. A ring R is $\mathcal{P}$-noetherian if $\mathcal{P}(\mathcal{I}_{\mathsf{R}}, \supseteq)$ holds.

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Let $\mathcal{P}$ be an acceptable property. A ring R is $\mathcal{P}$-noetherian if $\mathcal{P}(\mathcal{I}_R, \supseteq)$ holds.

If R is a coherent, strongly discrete and $\mathcal{P}$-Noetherian ring, so is $R[X]$.

The ideas of the proof

Let M be a coherent R-module and N a R-submodule of M. There is an increasing map from $\mathcal{J}_{\mathsf{M}}$ to $\mathcal{J}_{\mathsf{M}/\mathsf{N}} \times \mathcal{J}_{\mathsf{N}}$ (ordered by the product order).

For all $\mathsf{I} \in \mathcal{J}_{\mathsf{R}[\mathsf{X}]}$ we define $n(\mathsf{I})$ as the smallest integer such that $\mathsf{I} \cap \mathsf{R}[\mathsf{X}]_{n(\mathsf{I})}$ generates I as an ideal.

Let Θ be the following map

$$\begin{array}{rccc} \Theta \ : & \mathcal{J}_{\mathsf{R}[\mathsf{X}]} & \longrightarrow & \mathcal{J}_{\mathsf{R}} \times \overleftarrow{\sum_{n \geq 1}} \mathcal{J}_{\mathsf{R}[\mathsf{X}]_n} \\ & \mathsf{I} & \longmapsto & \left(\mathrm{LC}(\mathsf{I}), \mathsf{I} \cap \mathsf{R}[\mathsf{X}]_{n(\mathsf{I})}\right) \ . \end{array}$$

This is a decreasing map – the value set being ordered lexicographically.