# An Introduction to Control Theory

Alban Quadrat[1]

[1]INRIA Sophia Antipolis, CAFE Project, 2004 route des lucioles, BP 93, 06902 Sophia Antipolis cedex, France.

Alban.Quadrat@sophia.inria.fr

www-sop.inria.fr/cafe/Alban.Quadrat/index.html

# Purpose of the tutorial

- **I do not know why you are locked up with me for a 4h tutorial in control theory!** I only have some hints.

- At least, I know how the story started...

- The purposes of the tutorial are to:

  1. Give a **short introduction** to control theory.
  2. Show that some **connections exist between control theory and commutative algebra** (Lombardi, Coquand, Quitte...):

     Fractional ideals, lattices, projective/stably free/free modules, Prüfer/Bézout domains, coherent rings, projective free rings, stable range, minimal generating systems...

# Plan of the tutorial

- The **plan of the tutorial** is:

  1. Single-input single-ouput systems:

     **An introduction to the fractional ideal approach to stabilization problems**

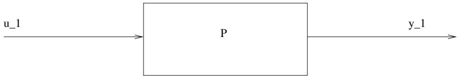
  2. Multi-input multi-ouput systems:

     **An introduction to the lattice approach to stabilization problems**

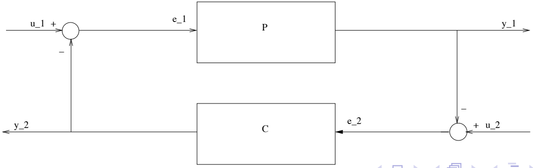
# Control theory

- **Control theory** can be divided into 3 main steps:

  1. **Modeling problems**: find a correct mathematical model for a real system coming from mechanics, electrical engineering, mathematical physics, biology...
  2. **Analysis problems**: analysis of the properties of the system (controllability, observability, stabilizability...).




  3. **Synthesis problems**: construction of a feedback controller which stabilizes and optimizes the performances of the closed-loop system, study the robustness issues...



# History & assumptions

- **History of control theory**:
  1. Prehistory: Watt (1769), Maxwell (1868), Lyapunov (1907),
  2. Frequency-domain approach (Black, Nyquist, Bode, 1930-40),
  3. Time-domain approach (Bellman, Pontryagin, Kalman, 1957-60): state-space systems, controllability, observability, optimal control, Kalman filter...
  4. Robust control (Zames, Desoer, Francis, Doyle, 1980-90), $\mu$-synthesis, Linear Matrix Inequalities (LMIs)...
  5. Future?
- We shall only study **time-invariant linear systems** defined by:
  1. ordinary or partial differential equations,
  2. differential time-delay equations.
- We shall focus on

  **synthesis problems** within a **frequency-domain approach**.

# An introduction to the fractional ideal approach to stabilization problems

1. Linear control systems
2. Laplace transform
3. Transfer function
4. Signal spaces and algebras
5. Stability
6. Fractional representation approach
7. Analysis problems
8. Synthesis problems
9. Theory of fractional ideals
10. NSC for internal/strong/robust stabilizability
11. Parametrization of all stabilizing controllers

# Linear control systems

1. **Finite-dimensional linear systems:**

$$\dot{z}(t) = z(t) + u(t), \quad z(0) = 0, \quad y(t) = z(t).$$

2. **Infinite-dimensional linear systems:**

   2.1 **Differential time-delay equations:** $h \in \mathbb{R}_+$, i.e., $h \geq 0$,

$$\begin{cases} \dot{z}(t) = z(t) + u(t), \quad z(0) = 0, \\ y(t) = \begin{cases} 0, & 0 \leq t < h, \\ z(t-h), & t \geq h. \end{cases} \end{cases}$$

   2.2 **Partial differential equations:**

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2 \dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \quad x \in [0, l], \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x}, t), \quad \overline{x} \in [0, l]. \end{cases}$$

# Laplace transform

$$L_1(\mathbb{R}_+) = \{f \in \mathbb{R}_+ \to \mathbb{R} \mid \ \| f \|_1 = \int_0^{+\infty} |f(t)|\, dt < +\infty\}.$$

- **Definition:** Let $f \in \mathbb{R}_+ = [0, +\infty[ \to \mathbb{R}$ be a function such that:

$$\exists\, \alpha \in \mathbb{R} : \quad e^{-\alpha\, t} f \in L_1(\mathbb{R}_+).$$

Then, the **Laplace transform** of $f$ is defined by:

$$\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-s\, t} f(t)\, dt, \quad \forall\, s \in \mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha\}.$$

- **Notation:** We also denote $\mathcal{L}(f)$ by $\widehat{f}$.

- **Example:** $Y = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0 \end{cases}, \quad \mathcal{L}(Y) = \frac{1}{s}, \quad \mathcal{L}(\delta) = 1,$

$$\mathcal{L}(t^n\, Y) = \frac{(n+1)!}{s^{n+1}}, \quad \mathcal{L}(t^n\, e^{-\lambda\, t}\, Y) = \frac{(n+1)!}{(s+\lambda)^{n+1}},$$

$$\mathcal{L}(e^{-\lambda\, t} \cos(\omega\, t)\, Y) = \frac{(s+\lambda)}{(s+\lambda)^2 + \omega^2}.$$

# Properties

- **Theorem**: If $f$ is **Laplace transformable**, then we have:

  1. $\widehat{f}$ is analytic and bounded in $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha\}$.
  2. If $g$ is a **Laplace transformable function** such that $\widehat{f}(s) = \widehat{g}(s)$ in $\mathbb{C}_\alpha$, for some $\alpha \in \mathbb{R}$, then $f = g$.
  3. If $(f \star g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$, $t \geq 0$, then:
  $$\widehat{f \star g} = \widehat{f}\, \widehat{g}.$$
  4. If $g(t) = \begin{cases} f(t-h), & t \geq h, \\ 0, & 0 < t < h, \end{cases}$, then $\widehat{g}(s) = e^{-h\,s}\, \widehat{f}(s)$.
  5. If $f$ is $n$ **times differentiable** for $t > 0$ and $f^{(1)}, \ldots, f^{(n)}$ are **Laplace transformable,** then:
  $$\widehat{f^{(n)}}(s) = s^n\, \widehat{f}(s) - \sum_{i=0}^{n-1} s^{n-i-1}\, f^{(i)}(0_+).$$

# Transfer functions

- **Ordinary differential equation:**

$$\dot{z}(t) = z(t) + u(t), \;\; z(0) = 0 \quad \Rightarrow \widehat{z}(s) = \frac{1}{(s-1)}\,\widehat{u}(s).$$

- **Differential time-delay equation:**

$$\begin{cases} \dot{z}(t) = z(t) + u(t), \;\; x(0) = 0, \\ y(t) = \begin{cases} 0, & 0 \le t < h, \\ z(t-h), & t \ge h, \end{cases} \end{cases} \Rightarrow \widehat{y}(s) = \frac{e^{-hs}}{(s-1)}\,\widehat{u}(s).$$

- **Wave equation:**

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2\,\dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x}, t), \end{cases} \Rightarrow \widehat{y}(s) = \frac{\left(e^{-\frac{\overline{x}}{a}s} - e^{-\frac{(2l-\overline{x})s}{a}}\right)}{\left(1 - e^{-\frac{2a}{l}s}\right)}\,\widehat{u}(s).$$

# Explicit computations

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2\,\dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \quad y(t) = z(\overline{x},t), \end{cases}$$

$$\frac{\partial^2 z}{\partial t^2}(x,t) - a^2\,\frac{\partial^2 z}{\partial x^2}(x,t) = 0 \quad \Rightarrow \tfrac{d^2\widehat{z}(x,s)}{dx^2} - \tfrac{s^2}{a^2}\,\widehat{z}(x,s) = 0,$$

$$\Rightarrow \widehat{z}(x,s) = A(s)\,e^{-\frac{s}{a}\,x} + B(s)\,e^{\frac{s}{a}\,x}.$$

$$\begin{cases} \widehat{z}(0,s) = \widehat{u}(s), \\ \widehat{z}(l,s) = 0, \end{cases} \Rightarrow \begin{cases} A(s) = \dfrac{1}{\left(1-e^{-\frac{2\,a}{l}\,s}\right)}\,\widehat{u}(s), \\ B(s) = -\dfrac{e^{-\frac{2\,a}{l}\,s}}{\left(1-e^{-\frac{2\,a}{l}\,s}\right)}\,\widehat{u}(s), \end{cases}$$

$$\Rightarrow \widehat{y}(s) = \widehat{z}(\overline{x},s) = \frac{\left(e^{-\frac{\overline{x}}{a}\,s} - e^{-\frac{(2\,l-\overline{x})\,s}{a}}\right)}{\left(1 - e^{-\frac{2\,a}{l}\,s}\right)}\,\widehat{u}(s).$$

# Transfer functions

- **Heat equation:**

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) - \lambda^2 \dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x},t), \end{cases} \Rightarrow \widehat{y}(s) = \frac{\left(e^{\lambda\,(l-\overline{x})\,\sqrt{s}} - e^{-\lambda\,(l-\overline{x})\,\sqrt{s}}\right)}{\left(e^{\lambda\,l\,\sqrt{s}} - e^{-\lambda\,l\,\sqrt{s}}\right)}\,\widehat{u}(s).$$

- **Telegraph equation:**

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2 \dfrac{\partial^2 z}{\partial x^2}(x,t) - k\,z(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad \lim_{x\to+\infty} z(x,t) = 0, \\ y(t) = z(\overline{x},t), \end{cases} \Rightarrow \widehat{y}(s) = e^{\frac{-\sqrt{s^2-k}}{a}\,\overline{x}}\,\widehat{u}(s).$$

# Transfer functions

- **Electric transmission line:**

$$\begin{cases} \dfrac{\partial V}{\partial x}(x,t) + L\,\dfrac{\partial I}{\partial t}(x,t) + R\,I(x,t) = 0, \\ \dfrac{\partial I}{\partial x}(x,t) + C\,\dfrac{\partial V}{\partial t}(x,t) + G\,V(x,t) = 0, \\ V(x,0) = 0, \quad I(x,0) = 0, \\ V(0,t) = u(t), \quad \lim_{x\to+\infty} V(x,t) = 0, \\ V(\overline{x},t) = y_1(t), \quad I(\overline{x},t) = y_2(t), \end{cases}$$

$$\Rightarrow \begin{cases} \widehat{y_1}(s) = e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}}\,\widehat{u}(s), \\ \widehat{y_2}(s) = \sqrt{\frac{C\,s+G}{L\,s+R}}\,e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}}\,\widehat{u}(s). \end{cases}$$

# Kernel representation (convolution)

- **Inverse Laplace transform**:

$$f(t) = \mathcal{L}^{-1}\left(\widehat{f}\right)(t) = \frac{1}{2\,\pi\,i}\int_{a-i\,\infty}^{a+i\,\infty} e^{s\,t}\,\widehat{f}(s)\,ds = f(t),\ a > \alpha,\ t > 0.$$

- **Ordinary differential equation:**

$$\widehat{y}(s) = \frac{1}{(s-1)}\,\widehat{u}(s) \Rightarrow y = \left(\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right) \star u,$$

$$\Rightarrow y(t) = \int_0^t e^{t-\tau}\,u(\tau)\,d\tau, \quad t \geq 0.$$

- **Differential time-delay equation:**

$$\widehat{y}(s) = \frac{e^{-h\,s}}{(s-1)}\,\widehat{u}(s) \Rightarrow y = \left(\mathcal{L}^{-1}\left(\frac{e^{-h\,s}}{s-1}\right)\right) \star u,$$

$$\Rightarrow y(t) = \int_0^{t-h} e^{t-h-\tau}\,u(\tau)\,d\tau, \quad t \geq h, \quad 0 \quad \text{else}.$$

# Kernel representation (convolution)

- **Wave equation:**

$$\widehat{y}(s) = \frac{\left(e^{-\frac{\overline{x}}{a}s} - e^{-\frac{(2\,l-\overline{x})\,s}{a}}\right)}{\left(1 - e^{-\frac{2\,a}{l}\,s}\right)}\,\widehat{u}(s)$$

$$\Rightarrow \widehat{y}(s) = \left(e^{-\frac{s}{a}\,\overline{x}}\ \sum_{n=0}^{+\infty} e^{-\frac{2\,n\,s}{a}\,l} - e^{-\frac{s}{a}\,(2\,l-\overline{x})}\ \sum_{n=0}^{+\infty} e^{-\frac{2\,n\,s}{a}\,l}\right)\,\widehat{u}(s),$$

$$\Rightarrow y(\overline{x},t) = \sum_{n=0}^{+\infty} u\left(t - \frac{2\,n\,l+\overline{x}}{a}\right) - \sum_{n=1}^{+\infty} u\left(t - \frac{2\,n\,l-\overline{x}}{a}\right).$$

- **Telegraph equation:** $k = \beta^2 > 0$.

$$\widehat{y}(s) = e^{\frac{-\sqrt{s^2-\beta^2}}{a}\,\overline{x}}\,\widehat{u}(s)$$

$$\Rightarrow y(\overline{x},t) = u\left(t - \frac{\overline{x}}{a}\right) + \beta\left(\frac{\overline{x}}{a}\right)\int_{\frac{\overline{x}}{a}}^{t} u(t-\tau)\,\frac{I_1\left(\beta\sqrt{\tau^2 - \left(\frac{\overline{x}}{a}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{\overline{x}}{a}\right)^2}}\,d\tau.$$

# Signal spaces

- Let us define the **right half plane** $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s > 0\}$.
- The **Hardy algebra** $H_\infty(\mathbb{C}_+)$ is defined by:

$$H_\infty(\mathbb{C}_+) = \{\text{analytic functions } f \text{ in } \mathbb{C}_+ \mid \ \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}.$$

$H_\infty(\mathbb{C}_+)$ is a commutative **Banach algebra**.

- The **Hardy vector-space** $H_2(\mathbb{C}_+)$ is defined by:

$$H_2(\mathbb{C}_+) = \{\,\text{analytic functions } f \text{ in } \mathbb{C}_+ \mid$$

$$\| f \|_2 = \sup_{x \in \mathbb{R}_+} \left( \int_{-\infty}^{+\infty} |f(x+iy)|^2 dy \right)^{1/2} < +\infty\}$$

$H_2(\mathbb{C}_+)$ is a **Hilbert space** and $H_2(\mathbb{C}_+) = \mathcal{L}(L_2(\mathbb{R}_+))$, where:

$$L_2(\mathbb{R}_+) = \{g : \mathbb{R}_+ \to \mathbb{R} \mid \left( \int_0^{+\infty} |g(t)|^2 \, dt \right)^{1/2} < +\infty\}.$$

# $L_2(\mathbb{R}_+) - L_2(\mathbb{R}_+)$-stability

- **Theorem**:

1. $\forall\, a,\, b \in H_\infty(\mathbb{C}_+),\ \forall\, f,\, g \in H_2(\mathbb{C}_+):\ a\,f + b\,g \in H_2(\mathbb{C}_+)$.
2. The **linear operator**

$$\begin{array}{rcl} \Lambda : H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ u & \longmapsto & h\,u, \end{array}$$

is **bounded**, i.e.:

$$\mathrm{dom}(\Lambda) = \{u \in H_2(\mathbb{C}_+) \mid \Lambda(u) \in H_2(\mathbb{C}_+)\} = H_2(\mathbb{C}_+),$$

iff $h \in H_\infty(\mathbb{C}_+)$. Then, we have:

$$\| \Lambda \|_{\mathcal{L}}\, (H_2(\mathbb{C}_+), H_2(\mathbb{C}_+)) = \sup_{0 \neq u \in H_2(\mathbb{C}_+)} \frac{\| \, h\, u \, \|_2}{\| \, u \, \|_2} = \| \, h \, \|_\infty \, .$$

# Example

- $p = \dfrac{1}{s-1} \notin H_\infty(\mathbb{C}_+)$ as $p$ has a **pole** at $1 \in \mathbb{C}_+$,

$$\Rightarrow \Lambda : \begin{array}{ccc} H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ \widehat{u} & \longmapsto & \widehat{y} = \dfrac{1}{(s-1)}\,\widehat{u}, \end{array} \quad \text{is } \textbf{unbounded},$$

$$\Rightarrow \lambda : \begin{array}{ccc} L_2(\mathbb{R}_+) & \longrightarrow & L_2(\mathbb{R}_+), \\ u & \longmapsto & y = e^t\, Y \star u, \end{array} \quad \text{is } \textbf{unbounded},$$

$$\mathrm{dom}\,\Lambda = \left(\tfrac{s-1}{s+1}\right) H_2(\mathbb{C}_+), \quad \mathrm{dom}\,\lambda = (\delta - 2\,e^{-t}\,Y) \star L_2(\mathbb{R}_+).$$

- $u = e^{-t}\,Y \in L_2(\mathbb{R}_+)$: $\quad \| u \|_2 = \dfrac{1}{\sqrt{2}}, \quad \widehat{u} = \dfrac{1}{s+1} \in H_2(\mathbb{C}_+)$,

$$\Rightarrow \widehat{y} = \frac{1}{s^2-1} \notin H_2(\mathbb{C}_+), \quad \frac{1}{s^2-1} = \mathcal{L}((\mathrm{sh}\, t)\, Y).$$

- $y(t) = \int_0^t e^{t-\tau}\, e^{-\tau}\, d\tau = (\mathrm{sh}\, t)\, Y \notin L_2(\mathbb{R}_+)$.

# Example

- $p = \dfrac{e^{-h\,s}}{s-1} \notin H_\infty(\mathbb{C}_+)$ as $p$ has a **pole** at $1 \in \mathbb{C}_+$,

$$\Rightarrow \Lambda : \begin{array}{rcl} H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ \widehat{u} & \longmapsto & \widehat{y} = \dfrac{e^{-h\,s}}{(s-1)}\,\widehat{u}, \end{array} \quad \text{is } \textbf{unbounded},$$

$$\Rightarrow \lambda : \begin{array}{rcl} L_2(\mathbb{R}_+) & \longrightarrow & L_2(\mathbb{R}_+), \\ u & \longmapsto & y = e^{t-h}\,Y \star u, \end{array} \quad \text{is } \textbf{unbounded}.$$

- $u = e^{-t}\,Y \in L_2(\mathbb{R}_+)$: $\quad \| \, u \, \|_2 = \dfrac{1}{\sqrt{2}}, \quad \widehat{u} = \dfrac{1}{s+1} \in H_2(\mathbb{C}_+),$

$$\Rightarrow \hat{y} = \frac{e^{-h\,s}}{s^2-1} \notin H_2(\mathbb{C}_+), \quad \frac{e^{-h\,s}}{s^2-1} = \mathcal{L}((\mathrm{sh}\,(t-h))\,Y).$$

- $y(t) = \int_0^{t-h} e^{t-h-\tau}\,e^{-\tau}\,d\tau = (\mathrm{sh}\,(t-h))\,Y \notin L_2(\mathbb{R}_+).$

# Example

- The **transfer function** $p = \frac{e^{-\frac{\overline{x}}{a}s} - e^{-\frac{(2l-\overline{x})}{a}s}}{1-e^{-\frac{2a}{l}s}}$ is such that

$\| p \|_\infty = +\infty$ as $p$ has **poles** at $s_k = \frac{l}{a}\pi k i,\ k \in \mathbb{Z}$

$$\Rightarrow p \textbf{ is not } H_2(\mathbb{C}_+) - H_2(\mathbb{C}_+)\textbf{-stable}.$$

- The **transfer function** $p = \frac{1}{s} \notin H_\infty(\mathbb{C}_+)$ as $\| p \|_\infty = +\infty$.

$$\Rightarrow \begin{array}{rcl} \Lambda : H_2(\mathbb{C}_+) & \longrightarrow & H_2(\mathbb{C}_+), \\ \widehat{u} & \longmapsto & \widehat{y} = \frac{1}{s}\,\widehat{u}, \end{array} \quad \text{is } \textbf{unbounded},$$

$$\Rightarrow \begin{array}{rcl} \lambda : L_2(\mathbb{R}_+) & \longrightarrow & L_2(\mathbb{R}_+), \\ u & \longmapsto & y(t) = \int_0^t u(\tau)\,d\tau, \end{array} \quad \text{is } \textbf{unbounded}.$$

$u(t) = Y/(t+1) \in L_2(\mathbb{R}_+)$, $\| u \|_2 = 1$, $\quad y = \ln(1+t) \notin L_2(\mathbb{R}_+)$.

# Signal spaces

- $L_1(\mathbb{R}_+) = \{f : [0, +\infty[ \to \mathbb{R} \mid \; \| f \|_1 = \int_0^{+\infty} |f(t)|\, dt < +\infty\}$,

$$l_1(\mathbb{Z}_+) = \{a : \mathbb{Z}_+ = \{0, 1, \ldots\} \to \mathbb{R} \mid \; \| (a_i)_{i \in \mathbb{Z}_+} \|_1 = \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

- **Definition:** The **Wiener algebra** $\mathcal{A}$ is defined by:

$$\mathcal{A} = \{f = g + \textstyle\sum_{i=0}^{+\infty} a_i\, \delta_{(t-h_i)} \mid \quad g \in L_1(\mathbb{R}_+),\ (a_i)_{i \in \mathbb{Z}_+} \in l_1(\mathbb{Z}_+),$$
$$0 = h_0 \leq h_1 \leq h_2 \leq \ldots\}.$$

- $\mathcal{A}$ is a commutative **Banach algebra** w.r.t.:

$$\| f \|_{\mathcal{A}} = \| g \|_1 + \| (a_i)_{i \in \mathbb{Z}_+} \|_1 \; .$$

- $\widehat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}, \quad \| \widehat{f} \|_{\widehat{\mathcal{A}}} = \| f \|_{\mathcal{A}}.$

## $L_\infty(\mathbb{R}_+) - L_\infty(\mathbb{R}_+)$-stability

- **Theorem:** Let $p \in [1, +\infty[$.
  1. $\forall\, a,\, b \in \mathcal{A}, \quad \forall\, f,\, g \in L_p(\mathbb{R}_+): \quad a \star f + b \star g \in L_p(\mathbb{R}_+)$.
  2. The **linear operator**

$$\begin{aligned} \Lambda : L_\infty(\mathbb{R}_+) &\longrightarrow L_\infty(\mathbb{R}_+), \\ u &\longmapsto h \star u, \end{aligned}$$

  is **bounded**, i.e., $\operatorname{dom} \Lambda = L_\infty(\mathbb{R}_+)$, iff $\widehat{h} \in \widehat{\mathcal{A}}$ and:

$$\| \Lambda \|_{\mathcal{L}(L_\infty(\mathbb{R}_+), L_\infty(\mathbb{R}_+))} = \sup_{0 \neq u \in L_\infty(\mathbb{R}_+)} \frac{\| h \star u \|_\infty}{\| u \|_\infty} = \| \hat{h} \|_{\widehat{\mathcal{A}}} .$$

  3. $\hat{f} \in \widehat{\mathcal{A}}$ is **analytic** and **bounded** in $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$ and **continuous** on $i\,\mathbb{R}$:

$$\| \widehat{f} \|_\infty \leq \| \widehat{f} \|_{\widehat{\mathcal{A}}}, \quad \widehat{\mathcal{A}} \subset H_\infty(\mathbb{C}_+) \quad (e^{-\frac{1}{s}} \in H_\infty(\mathbb{C}_+) \backslash \widehat{\mathcal{A}}).$$

  4. **BIBO-stability** $\Rightarrow$ $L_p(\mathbb{R}_+) - L_p(\mathbb{R}_+)$**-stability**.

# Examples

- $\frac{1}{s-1} \notin H_\infty(\mathbb{C}_+) \Rightarrow \frac{1}{s-1} \notin \widehat{\mathcal{A}} \quad (\widehat{\mathcal{A}} \subset H_\infty(\mathbb{C}_+))$.

Let $e^{-t}\,Y \in L_\infty(\mathbb{R}_+)$, $\quad \| \, e^{-t}\,Y \, \|_\infty = 1$. Then, we have:

$$y(t) = \int_0^t e^{t-\tau}\,e^{-\tau}\,d\tau = (\operatorname{sh} t)\,Y \notin L_\infty(\mathbb{R}_+).$$

- $\frac{e^{-h\,s}}{s-1} \notin H_\infty(\mathbb{C}_+) \Rightarrow \frac{e^{-h\,s}}{s-1} \notin \widehat{\mathcal{A}}$. Let us take $e^{-t}\,Y \in L_\infty(\mathbb{R}_+)$

$$\Rightarrow y(t) = \int_0^{t-h} e^{t-h-\tau}\,e^{-\tau}\,d\tau = (\operatorname{sh}(t-h))\,Y \notin L_\infty(\mathbb{R}_+).$$

- $p = \frac{e^{-\frac{\overline{x}}{a}\,s} - e^{-\frac{(2\,l-\overline{x})}{a}\,s}}{1-e^{-\frac{2\,a}{l}\,s}} \notin H_\infty(\mathbb{C}_+) \Rightarrow p \in \widetilde{\mathcal{A}}$, i.e.:

$$h = \sum_{n=0}^{+\infty} \delta_{\left(t-\frac{2\,n\,l+\overline{x}}{a}\right)} - \sum_{n=1}^{+\infty} \delta_{\left(t-\frac{2\,n\,l-\overline{x}}{a}\right)} \notin \mathcal{A}.$$

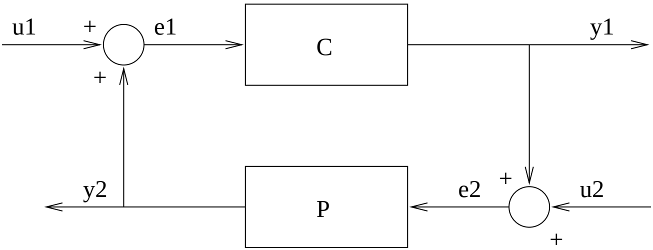
$\Rightarrow$ The 3 plants are **not BIBO stable**.

# Control theory

- Let the **open-loop** $\widehat{u} \longmapsto \widehat{y} = p\,\widehat{u}$ be **unstable**.

**Control theory: stabilization by feedback.**

- Is it possible to find a controller $c$ such that the **closed-loop is stable** $\forall\ \widehat{u}_1,\ \widehat{u}_2 \in H_2(\mathbb{C}_+)$ $(\forall\ u_1,\ u_1 \in L_\infty(\mathbb{R}_+))$?




- Can we **parametrize the set of stabilizing controllers of** $p$?

- Is it possible to find **robust/optimal controllers** $c$ of $p$?

# The fractional representation of plants

• (Zames) **The set of transfer functions has the structure of an algebra** (parallel $+$, serie $\circ$, proportional feedback . by $\mathbb{R}$).

• (Vidyasagar) **Let $A$ be an algebra of stable transfer functions** with a structure of an integral domain ($a\,b = 0,\ a \neq 0 \Rightarrow b = 0$) and its **the field of fractions**:

$$K = Q(A) = \{p = n/d \mid 0 \neq d,\, n \in A\}\,.$$

$\boxed{K \text{ represents the class of systems}}$

$p \in A \Rightarrow p$ is **stable**, $\quad p \in (K \backslash A) \Rightarrow p$ is **unstable**.

• (Zames) **The algebra $A$ of stable transfer functions has to be a normed algebra** so that we can consider the errors in the modelization & approximation of the real plant by a model

(e.g., $A$ is a **Banach algebra**:

$\| a\,b \|_A \leq \| a \|_A \| b \|_A, \quad \| 1 \|_A = 1$).

# Quotation

"...As soon as I read this, my immediate reaction was 'What is so difficult about handling that case? All one has to do is to write the unstable part as a ratio of two stable rational functions!' **Without exaggeration, I can say that the idea occurred to me within no more than 10 min. So there it is the best idea I have had in my entire research career, and it took less than 10 min. All the thousands of hours I have spent thinking about problems in control theory since have not resulted in any ideas as good as this one. I don't think I know what the 'moral of this story' really is !'**',

"...It turns out that this seemingly simple stratagem leads to conceptually simple and computationally tractable solutions to many important and interesting problems."

M. Vidyasagar, "A brief history of the graph topology", *European J. of Control*, 2 (1996), 80-87.

# Examples

- Let $RH_\infty = \mathbb{R}(s) \cap H_\infty(\mathbb{C}_+)$ be the **algebra of exponentially-stable finite-dimensional plants**, i.e.:

$$RH_\infty = \{n/d \in \mathbb{R}(s) \mid \deg n \leq \deg d,\ d(\bar{s}) = 0 \Rightarrow \mathrm{Re}\,\bar{s} < 0\}.$$

$$p = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \quad \frac{s-1}{s+1} \in RH_\infty \Rightarrow p \in Q(RH_\infty).$$

- $\hat{\mathcal{A}}$: **algebra of BIBO-stable $\infty$-dimensional plants**:

$$p = \frac{e^{-h\,s}}{s-1} = \frac{\left(\frac{e^{-h\,s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-h\,s}}{s+1}, \quad \frac{s-1}{s+1} \in \hat{\mathcal{A}} \Rightarrow p \in Q(\hat{\mathcal{A}}).$$

- $H_\infty(\mathbb{C}_+)$: **algebra of $L_2(\mathbb{R}_+)$-stable $\infty$-dimensional plants**:

$$p = \frac{(1 + e^{-2\,s})}{(1 - e^{-2\,s})} \in Q(H_\infty(\mathbb{C}_+)) : \quad 1 + e^{-2\,s},\ 1 - e^{-2\,s} \in H_\infty(\mathbb{C}_+).$$

# (Weakly) coprime factorization

- Let $A$ be an algebra of **stable transfer functions** and:

$$K = Q(A) = \{n/d,\ 0 \neq d,\ n \in A\}.$$

- **Definition**: A transfer function $p \in K$ is said to admit a **weakly coprime factorization** if:

$$\exists\, 0 \neq d,\, n \in A : \quad p = n/d, \quad \forall\, k \in K : \ k\, n,\ k\, d \in A \Rightarrow k \in A.$$

- **Definition**: A transfer function $p \in K$ is said to admit a **coprime factorization over** $A$ if:

$$\exists\, 0 \neq d,\, n,\, x,\, y \in A : \quad p = n/d, \quad d\, x - n\, y = 1.$$

- **A coprime factorization is a weakly coprime factorization**:

$$k \in K : \ k\, n,\ k\, d \in A \ \Rightarrow k = (k\, d)\, x - (k\, n)\, y \in A.$$

# Examples

- **Example**: Let $A = RH_\infty$ and $p = \frac{1}{(s-1)} \in \mathbb{R}(s)$. Then,

$$p = \frac{n}{d}, \quad n = \frac{1}{(s+1)\,(s+2)}, \quad d = \frac{(s-1)}{(s+1)\,(s+2)} \in A,$$

is **not a weakly coprime facorization** as:

$$(s+2) \in Q(A) = \mathbb{R}(s), \quad (s+2) \notin A, \quad \begin{cases} (s+2)\,n = \frac{1}{(s+1)} \in A, \\ (s+2)\,d = \frac{(s-1)}{(s+1)} \in A. \end{cases}$$

- **Example**: Let $A = RH_\infty$ and $p = \frac{1}{(s-1)} \in \mathbb{R}(s)$. Then,

$$p = \frac{n}{d}, \quad n = \frac{1}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)} \in A,$$

is a **coprime factorization** of $p$ as we have:

$$\frac{(s-1)}{(s+1)} - (-2)\,\frac{1}{(s+1)} = 1, \quad x = 1, \quad y = -2.$$

# Internal stabilizability

- Let $A$ be an algebra of **stable transfer functions**, $K = Q(A)$.
- Let $p \in K$ be a **plant** and $c \in K$ a **controler**.
- The **closed-loop system** is defined by:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \begin{cases} y_1 = e_2 - u_2, \\ y_2 = e_1 - u_1. \end{cases}$$

- **Definition**: $c$ **internally stabilizes** $p$ if we have:

$$H(p, c) = \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \in A^{2\times 2}.$$

$\Rightarrow$ $c$ is then called a **stabilizing controler** of $p$.

# Example

- **Example**: $A = RH_\infty, \quad K = Q(A) = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{(s-1)}, \\ c = -\frac{(s-1)}{(s+1)}, \end{cases} \Rightarrow \begin{cases} e_1 = \frac{(s+1)}{(2s+1)} \, u_1 + \frac{s(s+1)}{(2s+1)(s-1)} \, u_2, \\ e_2 = \frac{(-s+1)}{(2s+1)} \, u_1 + \frac{(s+1)}{(2s+1)} \, u_2. \end{cases}$$

$\Rightarrow$ $c$ **does not internally stabilize** $p$ because:

$$\frac{s(s+1)}{(2s+1)(s-1)} \notin RH_\infty \quad (\text{pole in } 1 \in \mathbb{C}_+).$$

- **Example**: $A = RH_\infty, \quad K = Q(A) = \mathbb{R}(s)$.

$$\begin{cases} p = \frac{s}{(s-1)}, \\ c = 2, \end{cases} \Rightarrow \begin{cases} e_1 = -\frac{(s-1)}{(s+1)} \, u_1 - \frac{s}{(s+1)} \, u_2, \\ e_2 = -2 \, \frac{(s-1)}{(s+1)} \, u_1 - \frac{(s-1)}{(s+1)} \, u_2. \end{cases}$$

$\Rightarrow$ $c$ **internally stabilizes** the plant $p$.

# Strong and simultaneous stabilizations

- Let $A$ be an algebra of **stable transfer functions**, $K = Q(A)$.

- **Definition**: $p \in K$ is **strongly stabilizable** if there exists a **stable controller** $c$, i.e., $c \in A$, which internally stabilizes $p$.

- **Definition**: The plants $p_1, \ldots, p_n \in K$ are **simultaneously stabilizable** if $\exists\ c \in K$ which internally stabilizes $p_1, \ldots, p_n$.

- **Interests of the strong stabilization**:

Safety, good ability to track reference inputs.

- **Interests of the simultaneous stabilization**:

The controller is designed to stabilize a family of plants, e.g.:

operating conditions, failed modes, loss of sensors/actuators, changes of operating points.

# Examples

- **Example**: Let $A = RH_\infty$. The plant $p = \frac{1}{(s-1)}$ is **strongly stabilized** by $c = -2 \in A$ as we have:

$$\frac{1}{1-p\,c} = \frac{(s-1)}{(s+1)}, \quad \frac{p}{1-p\,c} = \frac{1}{(s+1)}, \quad \frac{c}{1-p\,c} = -\frac{2\,(s-1)}{(s+1)}.$$

- **Example**: Let $A = H_\infty(\mathbb{C}_+)$. The plant $p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in K$ is **strongly stabilized** by $c = -1$ as we have:

$$\frac{1}{1-p\,c} = \frac{1-e^{-2\,s}}{2}, \; \frac{p}{1-p\,c} = \frac{1+e^{-2\,s}}{2}, \; \frac{c}{1-p\,c} = -\frac{1-e^{-2\,s}}{2}.$$

- **Example**: Let $A = RH_\infty$. The plants defined by

$$p_1 = \frac{1}{(s+1)}, \quad p_2 = \frac{2\,s}{(s-1)\,(s+1)},$$

are **simultaneously stabilized** by $c = -2\,\frac{(s+1)}{(s-1)}$.

# Robust stabilizability

- Let $A$ be a **Banach algebra** of stable transfer functions

$$(\text{e.g.,} \quad A = H_\infty(\mathbb{C}_+), \quad \widehat{\mathcal{A}}, \quad A(\mathbb{D}), \quad W_+).$$

- **Definition**: Let $c \in K = Q(A)$ be a stabilizing controller of $p \in K$. Then, $c$ **robustly stabilizes** $p$ if there exits $\epsilon > 0$ such that $c$ internally stabilizes one of the family of plants:

1. **Additive perturbations**:

$$B_1(p, \delta) = \{p + \delta \mid \forall\, \delta \in A, \ \| \, \delta \, \|_A < \ \epsilon\}.$$

2. **Multiplicative perturbations**:

$$B_2(p, \delta) = \{p/(1 + \delta\, p) \mid \forall\, \delta \in A, \ \| \, \delta \, \|_A < \ \epsilon\}.$$

3. **Relative additive perturbations**:

$$B_3(p, \delta) = \{p\,(1 + \delta) \mid \forall\, \delta \in A, \ \| \, \delta \, \|_A < \ \epsilon\}.$$

4. **Relative multiplicative perturbations**:

$$B_4(p, \delta) = \{p/(1 + \delta) \mid \forall\, \delta \in A, \ \| \, \delta \, \|_A < \ \epsilon\}.$$

# Theory of fractional ideals

- Let $A$ be an integral domain and $K = \{n/d \,|\, 0 \neq d,\, n \in A\}$.
- **Definition**: A **fractional ideal** $J$ **of** $A$ is an $A$-submodule of $K$

$$(\forall\, a_1,\, a_2 \in A, \quad \forall\, m_1,\, m_2 \in J : \quad a_1\, m_1 + a_2\, m_2 \in J),$$

such that $\exists\, 0 \neq d \in A$ satisfying:

$$(d)\, J \triangleq \{a\, d \,|\, a \in J\} \subseteq A.$$

- **Example**: Let $A$ be an algebra of stable transfer functions and $p \in K = Q(A)$ a **transfer function**. Then,

$$J = (1,\ p) \triangleq A + A\, p$$

is a **fractional ideal** of $A$ as:

$$\exists\, 0 \neq d,\, n \in A : \ p = n/d \ \Rightarrow \ (d)\, J = A\, d + A\, n \subseteq A.$$

- $y = p\, u \Rightarrow (1,\ -p)\, (y \quad u)^T = 0 \Rightarrow J = (1,\ -p) = (1,\ p).$

# Theory of fractional ideals

- **Definition**: A fractional ideal $J$ of $A$ is **integral** if $J \subseteq A$.
- **Example**: If $p \in A$, then $J = (1,\ p) = A$. Conversely,

$$J = (1,\ p) = (1) \ \Rightarrow \ \exists\, n \in A : \quad p = n\,1 = n \in A.$$

$\Rightarrow$ the transfer function $p$ is **stable** iff $J = (1,\ p) = A$.

- **Definition**: A fractional ideal $J$ of $A$ is **principal** if $\exists\, k \in K$ :

$$J = (k) \triangleq A\,k = \{a\,k \,|\, a \in A\}.$$

- **Example**: $J = (1,\ p)$ is principal iff there exists $0 \neq k \in K$ such that $J = (k)$, i.e., iff there exist $0 \neq d,\ n,\ x,\ y \in A$ s.t.:

$$\begin{cases} 1 = d\,k, \\ p = n\,k, \\ k = x - y\,p \end{cases} \Leftrightarrow \begin{cases} k = 1/d, \\ p = n/d, \\ 1/d = x - y\,(n/d), \end{cases} \Leftrightarrow \begin{cases} p = n/d, \\ d\,x - n\,y = 1. \end{cases}$$

$\Rightarrow$ the transfer function $p$ admits a **coprime factorization** $p = n/d$ iff $J = (1/d)$, i.e., $J$ is **principal**.

# Example

- Let $A = H_\infty(\mathbb{C}_+)$ and $p = \frac{e^{-s}}{(s-1)} \in K = Q(A)$.
- Let $J = (1,\, p)$ be the fractional ideal of $A$ defined by 1 and $p$.
- We have $J = \left(\frac{s+1}{s-1}\right)$ as we have:

$$\begin{cases} 1 = \left(\frac{s-1}{s+1}\right)\,\left(\frac{s+1}{s-1}\right), \\ \\ \frac{e^{-s}}{(s-1)} = \left(\frac{e^{-s}}{s+1}\right)\,\left(\frac{s+1}{s-1}\right), \\ \\ \frac{(s+1)}{(s-1)} = \left(1 + 2\,\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2\,e\,\frac{e^{-s}}{(s-1)} \qquad (\star). \end{cases}$$

$p = \frac{n}{d}$, $n = \frac{e^{-s}}{(s+1)}$, $d = \frac{(s-1)}{(s+1)}$, is a **coprime factorization** of $p$:

$$(\star) \Leftrightarrow \left(\frac{s-1}{s+1}\right)\,\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) - \left(\frac{e^{-s}}{s+1}\right)\,(-2\,e) = 1.$$

# Theory of fractional ideals

- **Proposition**: Let $\mathcal{F}(A)$ be the **set of non-zero fractional ideals** of $A$ and $I, J \in \mathcal{F}(A)$. Then, we have:

$$\begin{cases} I\,J = \{\sum_{\text{finite}} a_i\, b_i \mid a_i \in I,\; b_i \in J\} \in \mathcal{F}(A), \\ I : J = \{k \in K \mid (k)\, J \subseteq I\} \in \mathcal{F}(A). \end{cases}$$

- **Example**: Let $p \in K$ and $J = (1,\; p)$. Then, we have

$$A : J = \{k \in K \mid k,\; k\,p \in A\} = \{d \in A \mid d\,p \in A\}$$

is called the **ideal of the denominators** of $p$.

$p$ admits a **weakly coprime factorization** $p = n/d$ iff:

$$\begin{aligned} \exists\, 0 \neq d,\, n \in A : \quad & A : (d,\; n) = \{k \in K \mid k\,d,\; k\,n \in A\} = A, \\ & \Leftrightarrow A : ((d)\,(1,\; p)) = A \Leftrightarrow (A : J) : (d) = A \\ & \Leftrightarrow (d^{-1})\,(A : J) = A \Leftrightarrow A : J = (d). \end{aligned}$$

# Example

- Let $A$ be the Banach algebra of analytic functions in the unit disc $\mathbb{D}$ whose Taylor series converge absolutely, i.e.:

$$W_+ = \left\{ f(z) = \sum_{i=0}^{+\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty \right\}.$$

- $A$ is the algebra of the **BIBO-stable causal filters**.
- Let us consider the **transfer function** $p = e^{-\left(\frac{1+z}{1-z}\right)}$:

$$\left\{ \begin{array}{l} n = (1-z)^3\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \\ d = (1-z)^3 \in A, \end{array} \right. \quad \Rightarrow p = n/d \in Q(A).$$

- Let us consider the **fractional ideal** $J = (1,\, p)$ of $A$.
- The ideal $A : J = \{d \in A \mid d\, p \in A\}$ **is not finitely generated**.

See R. Mortini & M. Von Renteln, "Ideals in Wiener algebra", J. Austral. Math. Soc., 46 (1989), 220-228.

$\Rightarrow$ $p$ **does not admit a (weakly) coprime factorization**.

# Example

• The **disc algebra** $A(\mathbb{D})$ is the Banach algebra of holomophic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are continuous on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

• We have $n = (1-z)\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \quad d = (1-z) \in A,$

$$\Rightarrow p = n/d = e^{-\left(\frac{1+z}{1-z}\right)} \in Q(A), \quad J = (1,\, p).$$

• $A : J = \{d \in A \mid d\, p \in A\} = \{d \in A \mid d(1) = 0\}$ is a maximal ideal of $A$ which is **not finitely generated.**

See R. Mortini, "Finitely generated prime ideals in $H^\infty$ and $A(\mathbb{D})$", Math. Z., 191 (1986), 297-302.

$\Rightarrow$ $p$ **does not admit a (weakly) coprime factorization** and $p$ is **not internally stabilizable**.

# Theory of fractional ideals

- **Definition**: $J \in \mathcal{F}(A)$ is **invertible** if $\exists\, I \in \mathcal{F}(A)$:

$$I\,J = A.$$

- **Proposition**: If $J$ is an **invertible** fractional ideal of $A$, then:

$$I = A : J = \{k \in K \,|\, (k)\,J \subseteq A\}.$$

- If $J$ is an **invertible** fractional ideal of $A$, we then **denote** by:

$$I = J^{-1}.$$

- **Proposition**: If $J$ is invertible, then we have:

$$(J^{-1})^{-1} = J.$$

# Theory of fractional ideals

- Let $p \in K$ and $J = (1,\ p)$. If $J$ is **invertible**, then we have:

$$1 \in J\,(A : J) = (1,\ p)\,(\{d \in A \mid d\,p \in A\}) = \{\alpha + \beta\,p \mid \alpha,\ \beta \in A : J\}$$

$$\Leftrightarrow \exists\, a,\ b \in A : \begin{cases} a - b\,p = 1, \\ a\,p \in A, \quad b\,p \in A. \end{cases}$$

If $a \neq 0$, then $c = b/a \in K$ satisfies:

$$H(p,\ c) = \begin{pmatrix} \frac{1}{1 - p\,c} & \frac{p}{1 - p\,c} \\ \frac{c}{1 - p\,c} & \frac{1}{1 - p\,c} \end{pmatrix} = \begin{pmatrix} a & a\,p \\ b & a \end{pmatrix} \in A^{2 \times 2},$$

$\Rightarrow c = b/a$ **internally stabilizes** $p$ $\quad (a = 0 \Rightarrow c = 1 - b$ **IS** $p)$.

- If $p$ is **internally stabilizable**, then there exists $c \in K$ s.t.:

$$a = \frac{1}{1 - p\,c} \in A, \quad a\,p = \frac{p}{1 - p\,c} \in A, \quad b = \frac{c}{1 - p\,c} \in A.$$

Let $I = (a,\ b)$. Then, $a - b\,p = 1 \in I\,J \Rightarrow I\,J = A \Rightarrow I = J^{-1}$.

# Example

- Let $A = H_\infty(\mathbb{C}_+)$, $\quad p = \frac{e^{-s}}{(s-1)} \in Q(A)$, $\quad J = (1,\, p)$:

$$\gcd\left(\frac{e^{-s}}{s+1},\, \frac{s-1}{s+1}\right) = 1 \;\Rightarrow\; A:\, J = \{d \in A \mid d\,p \in A\} = \left(\frac{s-1}{s+1}\right).$$

- $p$ is **internally stabilizable** iff $\exists\, a,\, b \in A : J$ s.t. $a - b\,p = 1$:

$$\Leftrightarrow \exists\, x,\; y \in A: \quad \begin{cases} a = \left(\frac{s-1}{s+1}\right)\, x, \\ b = \left(\frac{s-1}{s+1}\right)\, y, \\ a - b\,p = 1. \end{cases}$$

$$a - b\,p = 1 \Leftrightarrow \left(\frac{s-1}{s+1}\right)\,(x - p\,y) = 1 \quad \Leftrightarrow x = \frac{s+1}{s-1} + p\,y$$

$$\Leftrightarrow x = \frac{(s+1) + e^{-s}\,y}{s-1}$$

$$\Rightarrow ((s+1) + e^{-s}\,y(s))(1) = 0 \Rightarrow y(1) = -2\,e.$$

$$y(s) = -2\,e \Rightarrow x(s) = 1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) \in A.$$

# Example continued

- Therefore, we have:

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right) \left(1 + 2 \left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) \in A : J, \\ b = -2\, e \left(\frac{s-1}{s+1}\right) \in A :\ J, \\ a - b\, p = 1. \end{cases}$$

$\Rightarrow$ a **stabilizing controller** $c$ of $p$ is defined by:

$$c = \frac{b}{a} = -\frac{2\, e\, (s-1)}{(s-1) + 2\, (1 - e^{-(s-1)})} = -\frac{2\, e\, (s-1)}{s + 1 - 2\, e^{-(s-1)}}.$$

- $J = (1,\, p)$ is **invertible**, $J^{-1} = A : J = \left(\frac{s-1}{s+1}\right)$

$\Rightarrow J = (J^{-1})^{-1} = \left(\frac{s+1}{s-1}\right)$ is **principal** $\Rightarrow$ $p$ admits the **coprime factorization**:

$$p = \frac{n}{d}, \quad n = \frac{e^{-s}}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)}, \quad d\, x - n\, y = 1.$$

# SC for internal stabilizability

- Let $A$ be an algebra of stable transfer functions and $K = Q(A)$.
- Let $p \in K$ and $J = (1,\ p)$ a fractional ideal of $A$.
- $p$ admits a **coprime factorization** iff $J$ is **principal**.
- $p$ is **internally stabilizable** iff $J$ is a **invertible fractional ideal**.
- If $J = (k)$, $0 \neq k \in K$, then $J^{-1} = (1/k)$

$\Rightarrow$ **the existence of a coprime factorization is a sufficient condition for internal stabilizability.**

- $p = n/d$ is a coprime factorization, $d\,x - n\,y = 1$, $x \in A$, $y \in A$,

$$\Rightarrow \begin{cases} a = d\,x, \\ b = d\,y, \end{cases} \Rightarrow c = b/a = y/x \text{ is a } \textbf{stabilizing controller of } p.$$

$$(a - b\,p = d\,x - (d\,y)\,p = d\,x - n\,y = 1,$$

$$a,\ b \in A, \quad a\,p = n\,x \in A, \quad b\,p = n\,y \in A)$$

# Strong stabilizability

- $p$ is **strongly stabilizable** iff there exists $c \in A$ such that:

$$a = \frac{1}{1 - p\,c} \in A, \quad a\,p = \frac{p}{1 - p\,c} \in A, \quad b = \frac{c}{1 - p\,c} = c\,a \in A.$$

Using the fact that $c \in A$, we obtain:

$$J^{-1} = (a,\ b) = (a) = ((1 - p\,c)^{-1}) \Rightarrow J = (J^{-1})^{-1} = (1 - p\,c).$$

- We suppose that there exists $c \in A$ such that $(1,\ p) = (1 - p\,c)$

$$\Rightarrow \exists\, 0 \neq d,\ n \in A : \begin{cases} 1 = d\,(1 - p\,c), \\ p = n\,(1 - p\,c), \end{cases} \Rightarrow \begin{cases} p = n/d, \\ d - n\,c = 1, \end{cases}$$

$$\Rightarrow \left( \begin{array}{cc} 1 & -p \\ -c & 1 \end{array} \right)^{-1} = \left( \begin{array}{cc} d & n \\ d\,c & d \end{array} \right) \in A^{2 \times 2},$$

i.e., $c \in A$ internally stabilizes $p$, i.e., $p$ is **strongly stabilizable**.

# Example

- Let $A = H_\infty(\mathbb{C}_+)$, $K = Q(A)$, $p = \frac{(1+e^{-2s})}{(1-e^{-2s})} \in K$.
- We have $J = (1,\, p) = \left(\frac{1}{1-e^{-2s}}\right)$ because:

$$\begin{cases} 1 = (1 - e^{-2s})\, \frac{1}{(1-e^{-2s})}, \\ p = \dfrac{(1+e^{-2s})}{(1-e^{-2s})} = (1+e^{-2s})\, \dfrac{1}{(1-e^{-2s})}, \\ \dfrac{1}{(1-e^{-2s})} = \dfrac{1}{2} + \dfrac{1}{2}\, \dfrac{(1+e^{-2s})}{(1-e^{-2s})}. \end{cases}$$

$\Rightarrow$ **coprime factorization** $\begin{cases} p = \dfrac{(1+e^{-2s})}{(1-e^{-2s})}, \\ \frac{1}{2}\,(1-e^{-2s}) + \frac{1}{2}\,(1+e^{-2s}) = 1. \end{cases}$

$\Rightarrow$ $c = -1$ is a **stable stabilizing controller** of $p$.

- We check that $1 - p\, c = 1 + p = \frac{2}{(1-e^{-2s})}$

$$\Rightarrow J = (1,\, p) = (1/(1-e^{-2s})) = (1 - p\, c).$$

# Robust stabilization

- $c \in K = Q(A)$ **internally stabilizes** $p \in K$ iff:

$$(1,\ p)\,(1,\ c) = (1 - p\,c).$$

- Let $\delta \in A$. $c$ **internally stabilizes** $p$ and $p + \delta$ iff we have:

$$\begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ (1,\ p+\delta)\,(1,\ c) = (1 - (p+\delta)\,c), \end{cases} \Leftrightarrow \begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ (1,\ p)\,(1,\ c) = (1 - (p+\delta)\,c), \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ \left(\dfrac{1-(p+\delta)\,c}{1-p\,c}\right) = \left(1 - \dfrac{\delta\,c}{1-p\,c}\right) = A, \end{cases} \Leftrightarrow \begin{cases} c\ \textbf{IS}\ p, \\ 1 - \dfrac{(\delta\,c)}{(1-p\,c)} \in \mathrm{U}(A). \end{cases}$$

- If $A$ is a **Banach algebra**, then (**small gain theorem**):

$$\|\, 1 - a \,\|_A <\ 1 \Rightarrow a \in \mathrm{U}(A) = \{a \in A \mid \exists\, b \in A :\ a\,b = b\,a = 1\}.$$

$\Rightarrow$ **a sufficient condition for robust stabilization** ($c/(1 - p\,c) \in A$) is:

$$\|\, \delta \,\|_A <\ (\|\, c/(1-p\,c) \,\|_A)^{-1}.$$

# Robust stabilization

- Let $\delta \in A$. $c$ **internally stabilizes** $p$ and $p/(1+\delta\, p)$ iff we have:

$$\begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ \left(1,\ \frac{p}{(1+\delta\, p)}\right)\,(1,\ c) = \left(1 - \frac{p\,c}{(1+\delta\, p)}\right), \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ (1+\delta\, p,\ p)\,(1,\ c) = (1 - p\,c + \delta\, p), \end{cases}$$

$$\Leftrightarrow \begin{cases} (1,\ p)\,(1,\ c) = (1 - p\,c), \\ \left(\dfrac{1 - p\,c + \delta\, p}{1 - p\,c}\right) = \left(1 - \dfrac{\delta\, p}{1 - p\,c}\right) = A, \end{cases}$$

$$\Leftrightarrow \begin{cases} c \textbf{ IS } p, \\ 1 - \dfrac{(\delta\, p)}{(1 - p\,c)} \in \mathrm{U}(A). \end{cases}$$

$\Rightarrow$ **a sufficient condition for robust stabilization** $(p/(1-p\,c) \in A)$ is:

$$\|\, \delta \,\|_A < \ (\|\, p/(1-p\,c)\,\|_A)^{-1}.$$

# A few more results

- "**IS**" stands for **"internally stabilized/-zable"**.
- "**CF**" stands for **"coprime factorization"**.
- **Proposition**: Let $\delta \in A$, $p, c \in Q(A)$.

1. If $p$ is **IS** by $c$, then $p$ admits a **CF** $\Leftrightarrow$ $c$ admits a **CF**.
2. $p$ is **IS** and $p$ admits a **weakly CF** $\Leftrightarrow$ $p$ admits a **CF**.
3. $p$ is **IS** by $c \Leftrightarrow p + \delta$ is **IS** by $c/(1 + \delta\, c)$.
4. $p$ is **IS** by $c \Leftrightarrow p/(1 + \delta\, p)$ is **IS** by $c + \delta$.
5. $p$ is **IS** by $c \Leftrightarrow 1/p$ is **IS** by $1/c$.
6. $p$ is **externally stabilized** by $c$, i.e., $p\, c/(1 - p\, c) \in A$, iff:
$$(1,\ p\, c) = (1 - pc).$$
7. $p = n/d$ **CF**, $c = s/r$ **CF**. $p$ is **IS** by $c \Leftrightarrow d\, r - n\, s \in \mathrm{U}(A)$.

# Summary

- Let $A$ be a **ring of stable transfer functions** and $K = Q(A)$.
- Let $p \in K$ be a **transfer function**.
- Let $J = (1,\, p)$ be a **fractional ideal** of $A$ and:

$$A : J = \{d \in A \mid d\ p \in A\}.$$

- **Theorem**: 1. $p$ is **stable** iff $J = A$ iff $A : J = A$.

2. $p$ admits a **weakly coprime factorization** iff:

$$\exists\, 0 \neq d \in A : \quad A : J = (d).$$

Then, $p = n/d$, $(n = d\, p \in A)$, is a weakly coprime factorization.

3. $p$ is **internally stabilizable** iff $J$ is **invertible**, i.e., iff:

$$\exists\, a,\, b \in A, \quad a - b\, p = 1, \quad a\, p \in A.$$

If $a \neq 0$, then $c = b/a$ is a **stabilizing controller** of $p$ and:

$$J^{-1} = (a,\ b), \quad a = 1/(1 - p\, c), \quad b = c/(1 - p\, c).$$

# Summary

4. $c \in K$ **internally stabilizes** $p \in K$ if we have:

$$(1,\ p)\,(1,\ c) = (1 - p\,c).$$

5. $c \in K$ **externally stabilizes** $p \in K$ $(p\,c/(1 - p\,c) \in A)$ iff:

$$(1,\ p\,c) = (1 - p\,c).$$

6. $p$ is **strongly stabilizable** iff there exists $c \in A$ such that:

$$(1,\ p) = (1 - p\,c).$$

7. $p$ admits a **coprime factorization** iff $(1,\ p)$ is **principal**. Then, there exists $0 \neq d \in A$ such that

$$(1,\ p) = (1/d)$$

and $p = n/d$ is a **coprime factorization** of $p$ $\quad (n = d\,p \in A)$.

# Classification of the rings $A$

- **Theorem**: Let $A$ be a integral domain of stable transfer functions and $K = Q(A)$.

1. Every transfer function $p \in K$ admits a **weakly coprime factorization** iff $A$ is a **GCDD**, i.e., any two elements of $A$ admits a greatest common divisor.

2. Every transfer function $p \in K$ is **internally stabilizable** iff $A$ is a **Prüfer domain**, i.e., any f.g. ideal of $A$ is invertible.

3. Every transfer function $p \in K$ admits a **coprime factorization** iff $A$ is a **Bézout domain**, i.e., any f.g. ideal of $A$ is principal.

- $RH_\infty$ is a **PID** $\Rightarrow$ GCD, Prüfer and Bézout domains.
- $H_\infty(\mathbb{C}_+)$ is a **GCDD** but is not a Prüfer and a Bézout domain.

$$(\exists\, x,\, y \in H_\infty(\mathbb{C}_+) :\; d\,x - n\,y = 1 \Leftrightarrow \inf_{s \in \mathbb{C}_+}(|d(s)| + |n(s)|) > 0$$

$$\gcd(e^{-s},\, 1/(s+1)) = 1, \quad \inf_{s \in \mathbb{C}_+}(|e^{-s}| + |1/(s+1)|) = 0.)$$

- $\widehat{\mathcal{A}}$ ???

# Pre-Bézout rings

• **Definition**: An integral domain $A$ is a **pre-Bézout ring** if, for every $d,\, n \in A$ such that there exists a greatest common divisor $[d,\, n]$ of $d$ and $n$, then there exist $x,\, y \in A$ satsifying:

$$d\, x - n\, y = [d,\, n].$$

• **Example**: The **disc algebra** $A(\mathbb{D})$ is the Banach algebra of holomophic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are continuous on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then, $A(\mathbb{D})$ is a **pre-Bézout ring**.

• **Proposition**: Let $A$ be a pre-Bézout ring. Then, we have:

1. $p \in Q(A)$ admits a **weakly coprime factorization**.

$$\Longleftrightarrow$$

2. $p \in Q(A)$ admits a **coprime factorization**.

# Stable range

- **Definition**: A ring $A$ has a **stable range** of $A$ equals 1 if, for every $(d,\ n) \in A^{1\times 2}$ admitting a right-inverse $(x,\ -y)^T \in A^2$,

$$\text{i.e.,} \quad d\,x - n\,y = 1,$$

there exists $c \in A$ such that:

$$d - n\,c \in \mathrm{U}(A) = \{a \in A \mid \exists\ b \in A : \ a\,b = b\,a = 1\}.$$

- **Theorem**: Let $A$ be a integral domain of transfer functions and $K = Q(A)$. Then, every transfer function $p \in K$ which admits a **coprime factorization** is **strongly stabilizable** iff $\mathrm{sr}(A) = 1$.

- **Example**: The following Banach algebras

$$H_\infty(\mathbb{D}), \quad H_\infty(\mathbb{C}_+), \quad A(\mathbb{D}), \quad W_+, \quad L_\infty(i\,\mathbb{R}),$$

have a stable range equals to 1 ($\mathrm{sr}(RH_\infty) = 2$!)

(Treil 92, Jones/Marshall/Wolff 86, Rupp 90).

# $RH_\infty \subset \widehat{\mathcal{A}} \subset H_\infty(\mathbb{C}_+)$

" . . . The foregoing results about rational functions are so elegant that one can hardly resist the temptation to try to generalize them to non-rational functions. **But to what class of functions? Much attention has been devoted in the engineering literature to the identification of a class that is wide enough to encompass all the functions of physical interest and yet enjoys the structural properties that allow analysis of the robust stabilisation problem**",

N. Young, "Some function-theoretic issues in feedback stabilization", in *Holomorphy Spaces*, MSRI Publications 33, 1998, 337-349.

# Parametrizations of all stabilizing controllers

- **Theorem**: Let $c$ be a **stabilizing controller** of $p \in Q(A)$, $a = 1/(1 - p\,c)$, $b = c/(1 - p\,c)$ and $J = (1,\, p)$. Then, **all stabilizing controllers** of $p$ are

$$c(q_1,\, q_2) = \frac{b + a^2\, q_1 + b^2\, q_2}{a + a^2\, p\, q_1 + b^2\, p\, q_2}, \quad (\star)$$

where $q_1$ and $q_2$ any element of $A$: $\quad a + a^2\, p\, q_1 + b^2\, p\, q_2 \neq 0$.

1. $(\star)$ depends on **only one free parameter**

$$\Leftrightarrow p^2 \textbf{ admits a coprime factorization } p^2 = s/r.$$

2. If $p^2$ admits a **coprime factorization** $p^2 = s/r$,

$$(\star) \Leftrightarrow c(q) = \frac{b + r\, q}{a + r\, p\, q}, \quad \forall\, q \in A: \; a + r\, p\, q \neq 0.$$

3. If $p$ admits a **coprime factorization** $p = n/d$, $d\, x - n\, y = 1$:

$$(\star) \Leftrightarrow c(q) = \frac{y + d\, q}{x + n\, q}, \quad \forall\, q \in A: \; x + n\, q \neq 0.$$

# K. Mori, CDC 1999, 973-975

- Let $A = \mathbb{R}[x^2, x^3]$ be the ring of **discrete time delay systems without the unit delay**.
- $A$ is used for high-speed circuits, computer memory devices.
- $p = (1 - x^3)/(1 - x^2) \in Q(A), \quad J = (1, p)$.
- Using $(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4)$, we get

$$p = \frac{(1 - x^3)}{(1 - x^2)} = \frac{(1 + x^2 + x^4)}{(1 + x^3)}.$$

$A : J = (1 - x^2, 1 + x^3)$ **is not principal** because $(x + 1) \notin A$.

$\Rightarrow$ $p$ **does not admit a (weakly) coprime factorization.**

- As $A : J = (1 - x^2,\, 1 + x^3)$, we then get:

$$J\,(A : J) = (1 - x^2,\, 1 + x^3,\, 1 - x^3,\, 1 + x^2 + x^4).$$

- We have $(1 + x^3)/2 + (1 - x^3)/2 = 1 \in J\,(A : J)$

$$\Rightarrow \begin{cases} a = (1 + x^3)/2 \in A : J, \\ b = -(1 - x^2)/2 \in A : J, \\ a - b\,p = 1, \end{cases}$$

$$\Rightarrow c = b/a = -(1 - x^2)/(1 + x^3) \text{ **internally stabilizes** } p.$$

- $J^{-1} = (1 - x^2,\, 1 + x^3) \Rightarrow J^{-2} = ((1 - x^2)^2,\, (1 + x^3)^2)$.
- $(x + 1) \notin A \Rightarrow J^{-2}$ is **not principal ideal of** $A$.

$\Rightarrow$ **all stabilizing controllers** of $p$ have the form:

$$c(q_1, q_2) = \frac{-(1 - x^2) + (1 - x^2)^2\, q_1 + (1 + x^3)^2\, q_2}{(1 + x^3) + (1 - x^2)\,(1 - x^3)\, q_1 + (1 + x^3)\,(1 + x^2 + x^4)\, q_2},$$

for all $q_1,\ q_2 \in A$ such that the denominator exists.

# V. Anantharam, IEEE TAC 30 (1985), 1030-1031

- $A = \mathbb{Z}[i\sqrt{5}]$, $p = (1 + i\sqrt{5})/2 \in K = \mathbb{Q}(i\sqrt{5})$, $J = (1,\, p)$.
- Using $2 \times 3 = (1 + i\sqrt{5})\,(1 - i\sqrt{5}) = 6$, we get

$$p = (1 + i\sqrt{5})/2 = 3/(1 - i\sqrt{5}),$$

and $A : J = (2,\, 1 - i\sqrt{5})$ is not a principal ideal of $A$.

$\Rightarrow p$ **does not admit a (weakly) coprime factorization.**

- $J\,(A : J) = (2,\, 1 + i\sqrt{5},\, 1 - i\sqrt{5},\, 3) = A$ as we have

$$-2 + 3 = (-2) - (-1 + i\sqrt{5})\,p = 1,$$

$\Rightarrow c = (1 - i\sqrt{5})/2$ **internally stabilizes** $p$.

- $J^{-2} = (2,\, 1 - i\sqrt{5})^2 = (2)$

$$\Rightarrow c(q) = \frac{(1 - i\sqrt{5}) - 2\,q}{2 - (1 + i\sqrt{5})\,q}, \quad \forall\, q \in A.$$

# Example

- It is well-known that the unstable plant $p = e^{-s}/(s-1)$ is internally stabilized by the **distributed delay** controller:

$$c = -2\,e\,(s-1)/\left(s+1-2\,e^{-(s-1)}\right).$$

$$\begin{cases} a = \frac{1}{(1-p\,c)} = \frac{(s+1-2\,e^{-(s-1)})}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ b = \frac{c}{1-p\,c} = -\frac{2\,e\,(s-1)}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ a\,p = \frac{p}{(1-p\,c)} = \frac{e^{-s}}{(s+1)}\,\frac{(s+1-2\,e^{-(s-1)})}{(s-1)} \in H_\infty(\mathbb{C}_+). \end{cases}$$

- We obtain that **all stabilizing controllers** of $p$ have the form:

$$c(l) = \frac{-2\,e + l\,\frac{(s-1)}{(s+1)}}{1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right) + l\,\frac{e^{-s}}{(s+1)}},\; l = \left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right)^2 q_1 + 4\,e^2\,q_2.$$

- This is the **Youla-Kučera parametrization** obtained from the following coprime factorization $p = n/d$:

$$n = \frac{e^{-s}}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)}, \quad (-2\,e)\,n - \left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) d = 1.$$

# Example: Smith predictor

- Let us consider the **transfer function**:

$$p = p_0\, e^{-\tau\, s}, \quad p_0 \in RH_\infty, \quad \tau \in \mathbb{R}_+.$$

- $p \in A = H_\infty(\mathbb{C}_+) \Rightarrow p = n/d$, where $d = 1$ and $n = p$.

$\Rightarrow$ **the parametrization of all stabilizing controllers** of $p$ is:

$$c(q) = \frac{q}{1 + q\, p_0\, e^{-\tau\, s}}, \quad \forall\, q \in A.$$

$\Rightarrow$ Let $c_0 \in \mathbb{R}(s)$ be a **certain stabilizing controller** of $p_0$.

$$\Rightarrow q_\star = \frac{c_0}{1 - p_0\, c_0} \in RH_\infty \subset A.$$

$$\Rightarrow c(q_\star) = \frac{c_0}{1 + p_0\, c_0\, (e^{-\tau\, s} - 1)} = \frac{c_0}{1 - c_0\, (p_0 - p)}$$

internally stabilizes $p$ and is called **Smith predictor**. We have:

$$\frac{p\, c(q_\star)}{1 - p\, c(q_\star)} = \left(\frac{p_0\, c_0}{1 - p_0\, c_0}\right) e^{-\tau\, s}.$$

# Convexity of $H(p, c)$

- Let $c$ be a **stabilizing controller** of $p \in Q(A)$.
- **All stabilizing controllers** of $p$ are given by

$$c(q_1, q_2) = \frac{(1 - p\, c_*)\, c_* + q_1 + q_2\, {c_*}^2}{(1 - p\, c_*) + q_1\, p + q_2\, p\, {c_*}^2}$$

$$\forall\, q_1,\, q_2 \in A: \quad (1 - p\, c_*) + q_1\, p + q_2\, p\, {c_*}^2 \neq 0.$$

- The **closed-loop system**

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{becomes}:$$

$$\begin{pmatrix} \frac{1}{1-p\,c_*} + q_1\, \frac{p}{(1-p\,c_*)^2} + q_2\, \frac{p\,{c_*}^2}{(1-p\,c_*)^2} & \frac{c_*}{1-p\,c_*} + q_1\, \frac{1}{(1-p\,c_*)^2} + q_2\, \frac{{c_*}^2}{(1-p\,c_*)^2} \\ \frac{p}{1-p\,c_*} + q_1\, \frac{p^2}{(1-p\,c_*)^2} + q_2\, \frac{(p\,c_*)^2}{(1-p\,c_*)^2} & \frac{1}{1-p\,c_*} + q_1\, \frac{p}{(1-p\,c_*)^2} + q_2\, \frac{p\,{c_*}^2}{(1-p\,c_*)^2} \end{pmatrix}.$$

$$\Rightarrow \forall\, \lambda \in A: \; H(p,\, c(\lambda\, q_1 + (1 - \lambda)\, q_1',\, \lambda\, q_2 + (1 - \lambda)\, q_2'))$$

$$= \lambda\, H(p,\, c(q_1,\, q_2)) + (1 - \lambda)\, H(p,\, c(q_1',\, q_2')).$$

# Sensitivity minimization

- Let $A$ be a **Banach algebra** ($H_\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, $W_+ \ldots$).
- Let $c$ be a **stabilizing controller** of $p \in Q(A)$ and:

$$a = 1/(1 - p\,c), \quad b = c/(1 - p\,c) \in A.$$

- Let $w \in A$ be a **weighted function.** Then, we have:

$$\inf_{c \in \mathrm{Stab}(p)} \| \, w/(1 - p\,c) \, \|_A = \inf_{q_1,\, q_2 \in A} \| \, w\,(a + a^2\,p\,q_1 + b^2\,p\,q_2) \, \|_A \quad (\star)$$

$\Rightarrow$ $(\star)$ is now a **convex problem**.

- If $p = n/d$ is a **coprime factorization** of $p$, $d\,x - n\,y = 1$,

$$\Rightarrow a + a^2\,p\,q_1 + b^2\,p\,q_2 = d\,(x + q\,n).$$

$\forall \in A, \quad \exists\, q_1, \, q_2 \in A : \quad q = x^2\,q_1 + y^2\,q_2$, where:

$$q_1 = d^2\,(1 - 2\,n\,y)\,q, \quad q_2 = n^2\,(1 + 2\,d\,x)\,q.$$

$$(\star) \Leftrightarrow \inf_{q \in A} \| \, w\,d\,(x + n\,q) \, \|_A \, .$$

# Open questions

- What are the **algebraic properties** of $\widehat{\mathcal{A}}$ ?

- Let $\mathcal{I}(A)$ be the group of invertible fractional ideals of $A$ and $\mathcal{P}(A)$ the group of principal fractional ideals of $A$.

$$\Rightarrow \mathcal{C}(A) = \mathcal{I}(A)/\mathcal{P}(A)$$

is sometimes called the **Picard group** of $A$. **Question:** $\mathcal{C}(\hat{\mathcal{A}})$ ?

- Is it possible to develop a **theory of divisors** over $H_\infty(\mathbb{C}_+)$?

- Let $p_1, p_2 \in K = Q(A)$. **When do we have:**

$$(1,\, p_2) \cong (1,\, p_1) \Leftrightarrow \exists\, 0 \neq k \in K :\ (1,\, p_2) = (k)\,(1,\, p_1)\,?$$

- The **simultaneous stabilization problem** is open when $p_i \in Q(A)$, $i = 1, \ldots, n$, **do not admit coprime factorizations**:

$$\exists\, c \in Q(A) :\ (1,\, p_i)\,(1,\, c) = (1 - p_i\, c), \quad 1 \leq i \leq n\,?$$

- $A = \{f \in H_\infty(\mathbb{C}_+) \mid \overline{f(\overline{s})} = f(s),\ \forall\, s \in \mathbb{C}_+\}$.

**Question:** $\mathrm{sr}(A) = 2\,?$

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