A constructive framework for spectral sequences

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Abstract

Spectral Sequences are a useful tool in Algebraic Topology providing information on homology groups by successive approximations from the homology of appropriate associated complexes. However, they are not real algorithms except in exceptional cases. On the contrary, the effective homology method provides a real algorithm for the computation of homology groups of complicated spaces. In this poster, we explain how the effective homology method can also be used for the computation of spectral sequences. A set of programs has been developed, allowing the user to calculate the whole set of components of spectral sequences associated to filtered complexes.

1 Spectral Sequences

Definition. A *Spectral Sequence* $E = \{E^r, d^r\}$ is a family of $\mathbb{Z}$-bigraded modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$, $r = 1, 2, \ldots$, each provided with a differential $d^r = \{d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\}$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$, $r = 1, 2, \ldots$.

Remark. The key point of this definition is the fact that each E^{r+1} in the spectral sequence is (up to isomorphism) the bigraded homology module of the preceding (E^r, d^r). Therefore if we know the stage r in the spectral sequence (E^r, d^r) we can build the bigraded module at the stage $r+1$, E^{r+1}, but this cannot define the next differential d^{r+1} which therefore must be independently defined too.
In some cases, it is in fact a matter of computability: the higher differentials of the spectral sequence are mathematically defined, but their definition is not constructive, that is, the differentials are not computable with the provided information. In the case of spectral sequences associated to filtered complexes a formal expression for the groups $E^r_{p,q}$ is known (see [2]), but it is not sufficient to compute them because the subgroups that appear in the expression are not of finite type in most situations and therefore they cannot be represented in a machine and cannot be obtained in general, only in very simple cases.
This means that *a spectral sequence is not an algorithm* that a machine can compute automatically: at each level r some extra information is needed and obviously a machine is not "able" to obtain this information.

2 Effective Homology

The effective homology method, based on the concept of "object with effective homology", provides a real algorithm for the computation of homology groups in many common situations. Some basic ideas about effective homology (that can be found in [3]) are included in the following definitions:
Definition. A *reduction* ρ between two chain complexes A and B (denoted by $A \Rightarrow B$) is a triple $\rho = (f, g, h)$ $\;{}^{h}\circlearrowright A \underset{g}{\overset{f}{\rightleftarrows}} B$
such that: $fg = \mathrm{id}_B$; $gf + d_A h + h d_A = \mathrm{id}_A$; $fh = 0$; $hg = 0$; $hh = 0$.

Remark. If $A \Rightarrow B$, then $A = B \oplus C$, with C acyclic, which implies that $H_*(A) \cong H_*(B)$.

Definition. A *(strong chain) equivalence* between the complexes A and B ($A \Longleftrightarrow B$) is a triple (D, ρ, ρ') where D is a chain complex, $\rho = (D \Rightarrow A)$ and $\rho' = (D \Rightarrow B)$.

Definition. An *object with effective homology* is a triple (X, EC, ε) where EC is an effective chain complex (that is, a free chain complex whose groups in each degree are finitely generated, so an elementary algorithm can compute its homology groups) and $C(X) \overset{\varepsilon}{\Longleftrightarrow} EC$.

Obviously $H_*(X) \cong H_*(EC)$, which means that **it is possible to compute the homology groups of X by means of those of EC.**

3 The Kenzo program

Kenzo [1] is a symbolic computation system for Algebraic Topology, developed by Francis Sergeraert and some coworkers. It was written in the programming language Common Lisp and has succeeded in computing homology groups of some complicated spaces that had not been determined before.
How does Kenzo compute the homology groups of a complex? If the complex is effective, then they can be determined by means of elementary operations with differential matrices. If the complex is not effective, then it tries to use the effective homology theory.
For instance, it can compute the homology groups of the space $X = \Omega(\Omega(\Omega(P^\infty\mathbb{R}/P^3\mathbb{R}) \cup_4 e^4) \cup_2 e^2)$. For dimension $n = 5$, we obtain

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> (homology X 5)
Homology in dimension 5 :
Component Z/16Z
Component Z/8Z
...
Component Z/2Z
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which means $H_5 X = \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2^{23}$.

4 Main result

Theorem. *Let C be a filtered chain complex with effective homology (HC, ε), with $\varepsilon = (D, \rho, \rho')$, $\rho = (f, g, h)$, and $\rho' = (f', g', h')$. Let us suppose that filtrations are also defined in the chain complexes HC and D. If the maps f, f', g, and g' are morphisms of filtered complexes (i.e. compatible with the filtrations) and both homotopies h and h' have order $\leq t$ (i.e. $h(F_pD), h'(F_pD) \subset F_{p+t}D \quad \forall p \in \mathbb{Z}$), then*

$$E(C)^r_{p,q} \cong E(HC)^r_{p,q} \quad \forall r > t$$

This theorem explains the relation between spectral sequences and effective homology, and states that **we can use the effective homology method not only to obtain the homology groups of a complex, like the Kenzo program, but also to compute the associated spectral sequence**.

5 A new module for Kenzo

A new module for Kenzo (in Common Lisp) has been developed, allowing computations of spectral sequences of filtered complexes.

The new methods work in a way similar to the mechanism of Kenzo for computing homology groups: for effective complexes, the formal expression of the groups $E^r_{p,q}$ can be computed through elementary methods with integer matrices; otherwise, the effective homology is needed to compute the $E^r_{p,q}$ by means of an analogous spectral sequence deduced of an appropriate filtration on the associated effective complex.
This new module allows us to compute:

1. The groups $E^r_{p,q}$, with their generators.
2. The differential maps $d^r_{p,q}$.
3. The convergence level: the minimal r such that $E^r_{p,q} = E^\infty_{p,q}$.
4. The filtration of H_{p+q} through the $E^\infty_{p,q}$.

6 Examples

The programs developed allow us to compute the two most classical examples of spectral sequences: Serre and Eilenberg-Moore. In the case of the Serre spectral sequence (associated to a fibration $F \hookrightarrow E \rightarrow B$), it is not difficult to prove that it is isomorphic after level $t = 2$ to the spectral sequence of the effective complex of the total space of the fibration, $E = F \times_\tau B$. The Eilenberg-Moore spectral sequence associated to the loop space of a simplicial set, ΩX, is isomorphic to the spectral sequence of the corresponding effective complex after $t = 1$, that is, for every level.
We present here as an example the $E^\infty_{p,q}$ of the Eilenberg-Moore spectral sequence between the space $X = \Omega S^3 \cup_2 e^3$ and its loop space $\Omega X = \Omega(\Omega S^3 \cup_2 e^3)$. $\pi_2 S^3 = \mathbb{Z}$ so it makes sense to attach a 3-disk by a map of degree 2 of its boundary to ΩS^3. Up to our knowledge, this spectral sequence has not appeared in the literature.

q \ p	0	1	2	3	4	5	6	7	8
12					$\mathbb{Z}_2^5$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$		
11				0	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^4$			
10			$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$			
9		0	0	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$				
8	0	0	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$				
7	0	0	0	$\mathbb{Z}_2^2$					
6	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$					
5	0	0	$\mathbb{Z}_2$						
4	0	$\mathbb{Z}$	$\mathbb{Z}_2$						
3	0	0							
2	0	$\mathbb{Z}_2$							
1	0								
0	$\mathbb{Z}$								

References

[1] X. Dousson, J. Rubio, F. Sergeraert, and Y. Siret, *The Kenzo program* (Institut Fourier, Grenoble, 1999) `http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`.

[2] S. Mac Lane, *Homology* (Springer, 1963).

[3] J. Rubio and F. Sergeraert. Constructive Algebraic Topology, *Bulletin des Sciences Mathématiques* **126** (2002) 389-412.