Problems as Solutions

Unique versus Constructive Existence

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. . . *If a question can be put at all, then it* can *also be answered.*

L. Wittgenstein [*TLP* 6.5]

Overview

Introduction and Folklore
Without Countable Choice
Why the Fan Theorem?
From Brouwer to Gödel

Situation:

(S, d) metric space, usually complete or even compact

$F : S \to \mathbb{R}_{\geqslant 0}$ continuous, mostly uniformly continuous

Problem:

If $\inf_{x \in S} F(x) = 0$, is there $\xi \in S$ with $F(\xi) = 0$?

Alternative formulation:

$$\forall \varepsilon > 0\, \exists x \in S\ \ (F(x) < \varepsilon) \quad \overset{?}{\Rightarrow} \quad \exists \xi \in S\, \forall \varepsilon > 0\ \ (F(\xi) < \varepsilon)$$

approximate solutions $\overset{?}{\rightsquigarrow}$ exact solution

First attempt:

Choose a sequence (x_n) in S with $F(x_n) < 1/n$ for all n.

In general this is an invocation of countable choice.

If S is compact, then (x_n) has a cluster point ξ in S, for which $F(\xi) = 0$.

We thus have used the Bolzano–Weierstraß theorem

BWT *Every sequence in a compact space has a cluster point,*

as one does in the usual proof of the minimum theorem

MIN *Every continuous function on a compact space has a minimum,*

by which we would have solved our problem anyway.

Both BWT and MIN, however, are essentially non–constructive.

There is a construction of $\inf_{x \in S} F(x)$ if F is uniformly continuous, and if S is compact: that is, totally bounded and complete.

So when does our problem still have a constructive solution?

In other words, when does a uniformly continuous function on a compact space have a minimum: that is, attain its infimum?

We say that F has *uniformly at most one* minimum if

$$\forall\delta\,\exists\varepsilon\,\forall x,y\;(F(x)<\varepsilon\wedge F(y)<\varepsilon\;\Rightarrow\;d(x,y)<\delta)$$

where δ, ε are positive rationals and x, y points of S.

If F has uniformly at most one minimum, then F has a minimum.

In fact, the sequence (x_n) in S with $F(x_n)<1/n$ is a Cauchy sequence. Hence if S is complete, then (x_n) has a limit ξ in S, for which $F(\xi)=0$.

One only needs that S is complete, and F sequentially continuous.

This was used intensively by U. Kohlenbach in the early 1990s. He could even start from a so–called modulus of uniqueness, which gives a modulus of convergence for the sequence (x_n).

It is a constructive solution, but a priori requires countable choice, by which the given data *is* an element of the completion of S.

We can do the same without choice following F. Richman, who considers the completion of S as the set $\widehat{S}$ of all locations on S.

A *location* on S is an $f : S \to \mathbb{R}$ with $\inf f = 0$ so that

$$|f(x) - f(y)| \leqslant d(x, y) \leqslant f(x) + f(y) \ .$$

The metric on the set $\widehat{S}$ of all locations on S is defined by

$$d(f, g) = \sup |f - g| = \inf (f + g) \ .$$

There is the isometric embedding

$$S \hookrightarrow \widehat{S}\,, \ z \mapsto d(z, \cdot\,) \ ,$$

with which S is dense in $\widehat{S}$, and $\widehat{S}$ is complete: that is, $\widehat{S} = \widehat{\widehat{S}}$.

Note that $d(f, z) = f(z)$ for all $f \in \widehat{S}$ and $z \in S$.

Every $f \in \widehat{S}$ is uniformly continuous, has uniformly at most one minimum, and satisfies

$$f(y) = \lim_{f(x) \to 0} d(x, y) \ .$$

The locations on $\mathbb{Q}$ form the Dedekind reals $\mathbb{R}$, for which $\mathbb{R} = \widehat{\mathbb{R}}$.

Each uniformly continuous $\varphi : S \to T$ extends uniquely to a

$$\widehat{\varphi} : \widehat{S} \to \widehat{T} \quad \text{with} \quad \widehat{\varphi}(f)(y) = \lim_{f(x) \to 0} d(\varphi(x), y)$$

which is uniformly continuous.

With all this, we can solve our problem without countable choice.

<u>Theorem 1</u> *If* $F : S \to \mathbb{R}$ *with* $\inf F = 0$ *is uniformly continuous and has uniformly at most one minimum, then*

$$f(y) = \lim_{F(x)\to 0} d(x, y)$$

defines an $f \in \widehat{S}$ *with* $\widehat{F}(f) = 0$ *for* $\widehat{F} : \widehat{S} \to \mathbb{R}$*.*

$$\left[\; d\left(\widehat{F}(f), 0\right) \stackrel{\checkmark}{=} \widehat{F}(f)(0) \stackrel{\checkmark}{=} \lim_{f(x)\to 0} \overbrace{d(F(x), 0)}^{F(x)} \stackrel{(!)}{=} 0 \;\right]$$

If $S = \mathbb{R}$ and $F(x) = |x - \xi|^k$ with $k \geqslant 1$ and $\xi \in \mathbb{R}$, then

$$f(y) = \lim_{x\to\xi} d(x, y) = d(\xi, y) = |y - \xi| \,.$$

If F is already a location, then $f = F$.

Brouwer's fan theorem can equivalently be put as

FAN *Every decidable binary tree without infinite path is finite.*

This is the contrapositive of weak König's lemma

WKL *Every infinite decidable binary tree has an infinite path,*

which is essentially non–constructive.

The logical form of FAN and WKL is

FAN $\quad \forall\alpha\,\exists n\; P(\overline{\alpha}n) \;\Rightarrow\; \exists n\,\forall\alpha\; P(\overline{\alpha}n)$

WKL $\quad \forall n\,\exists\alpha\; P(\overline{\alpha}n) \;\Rightarrow\; \exists\alpha\,\forall n\; P(\overline{\alpha}n)$

where $n \in \mathbb{N}$ and P is a decidable property of the finite initial segments

$$\overline{\alpha}n = (\alpha(0), \ldots, \alpha(n-1))$$

of infinite binary sequences α: that is, the points of the Cantor space.

We say that F as before has *at most one* minimum if

$$\forall x, y \, (d(x,y) > 0 \Rightarrow F(x) > 0 \vee F(y) > 0) \, .$$

With J. Berger and D. Bridges we found out in 2003 that

MIN! *Every uniformly continuous function on a compact space that has at most one minimum possesses a minimum*

is equivalent to FAN, but it was unclear why FAN has to do with this.

In 2005 J. Berger and H. Ishihara carried this over to other cases: e.g., with Brouwer's fixed point theorem in place of the minimum theorem.

All these equivalents of FAN have the form

> *If a problem on a compact space has approximate solutions and at most one solution, then it has an exact solution.*

It was still unclear why FAN occurred in this context.

The positivity principle

POS *Every uniformly continuous function on a compact space that attains only positive values has a positive infimum*

is the equivalent of FAN that was needed to prove MIN!.

With FAN in the form of POS we now know why it occurred there.

Theorem 2 FAN *is equivalent to*

UAM *If a uniformly continuous function on a compact space has at most one minimum, then it has uniformly at most one minimum.*

This sharpens FAN $\Rightarrow$ MIN!, because UAM $\Rightarrow$ MIN!.

The implications MIN! $\Rightarrow$ FAN and FAN $\Rightarrow$ POS have been proved before by J. Berger, D. Bridges, H. Ishihara, and the author.

To show the missing link POS $\Rightarrow$ UAM, we assume that S is compact, and $F : S \to \mathbb{R}_{\geqslant 0}$ uniformly continuous.

Note first that $d(x,y) \geqslant \delta$ defines a compact subset of $S \times S$ for all but countably many values of δ.

The condition "F has at most one minimum" can be put as

$$\forall x, y\ (\exists \delta\, d(x,y) \geqslant \delta \;\Rightarrow\; F(x) + F(y) > 0)\ ,$$

which is equivalent to

$$\forall \delta\, \forall x, y\ (d(x,y) \geqslant \delta \;\Rightarrow\; F(x) + F(y) > 0)\ .$$

By POS, this implies

$$\forall \delta\, \exists \varepsilon\, \forall x, y\ (d(x,y) \geqslant \delta \;\Rightarrow\; F(x) + F(y) \geqslant \varepsilon)\ ,$$

which is equivalent to "F has uniformly at most one minimum".

A *modulus of uniqueness* is a function $\mu : \delta \mapsto \varepsilon$ with

$$\forall \delta \, \forall x, y \; (F(x) < \mu(\delta) \wedge F(y) < \mu(\delta) \; \Rightarrow \; d(x,y) < \delta) \; .$$

Such a μ can be obtained from the corresponding $\forall\delta\,\exists\varepsilon$–statement "$F$ has uniformly at most one minimum" by countable choice.

<u>Corollary</u> *In the presence of countable choice,* FAN *is equivalent to*

MUN *If a uniformly continuous function on a compact space has at most one minimum, then it has a modulus of uniqueness.*

For S bounded, U. Kohlenbach obtained a modulus of uniqueness from

$$(*) \qquad \forall x, y \, (F(x) = 0 \wedge F(y) = 0 \Rightarrow d(x, y) = 0)$$

by his monotone functional interpretation (m.f.i.) going back to Gödel.

Condition $(*)$ is the contrapositive of "F has at most one minimum", and has the same m.f.i. whenever S is bounded.

In the presence of countable choice, FAN is thus equivalent to an instance of the m.f.i.: namely, to MUN.