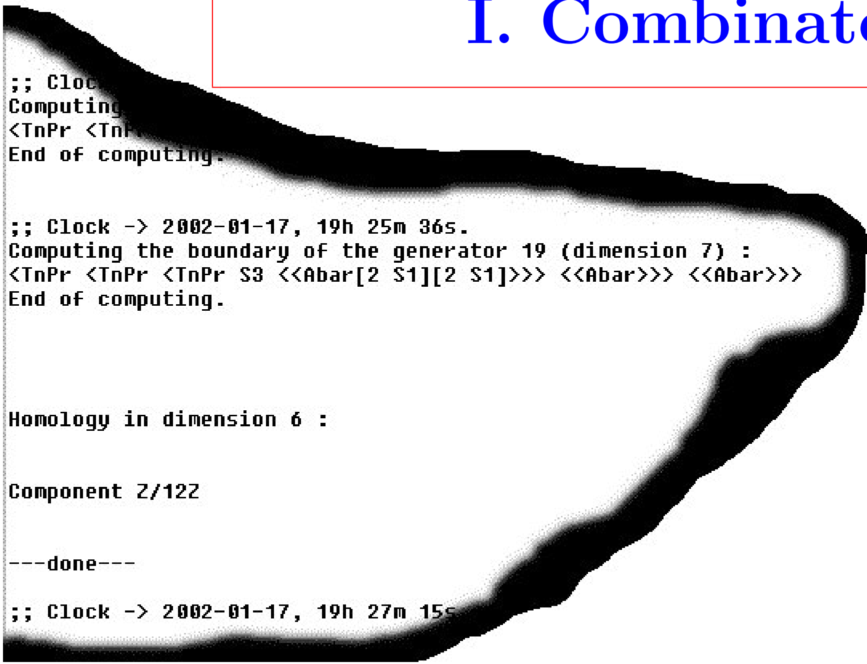
Algebraic Topology

(Castro-Urdiales tutorial)

I. Combinatorial Topology



Francis Sergeraert, Institut Fourier, Grenoble, France
Castro-Urdiales, January 9-13, 2006

Tutorial plan:

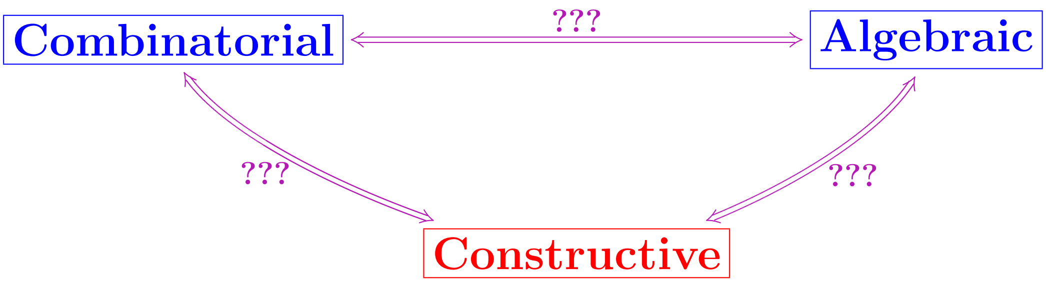
1. **Combinatorial Topology.**

2. **Homological Algebra.**

3. **Constructive Algebraic Topology.**

4. **Implemented Algebraic Topology.**

Essential terminological problem:



“General” topological spaces

cannot be directly installed in a computer.

A combinatorial translation is necessary.

Main methods:

1. **Simplicial complexes.**

2. **Simplicial sets.**

Warning: **Simplicial sets much more complex (!)**

but much more powerful than simplicial complexes.

Simplicial complex $K = (V, S)$ where:

1. V = set = set of vertices of K;

2. $S \in \mathcal{P}(\mathcal{P}_*^f(V))$ (= set of simplices) satisfying:

   (a) $\sigma \in S \Rightarrow \sigma$ = non-empty finite set of vertices;

   (b) $\{v\} \in S$ for all $v \in V$;

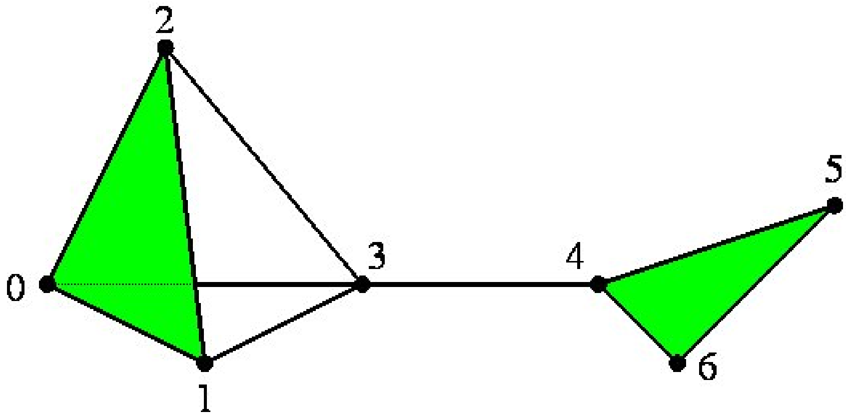
   (c) $\{(\sigma \in S)$ and $(\emptyset \neq \sigma' \subset \sigma)\} \Rightarrow (\sigma' \in S)$.

Notes:

1. V may be infinite ($\Rightarrow S$ infinite).

2. $\forall \sigma \in S$, σ is finite.

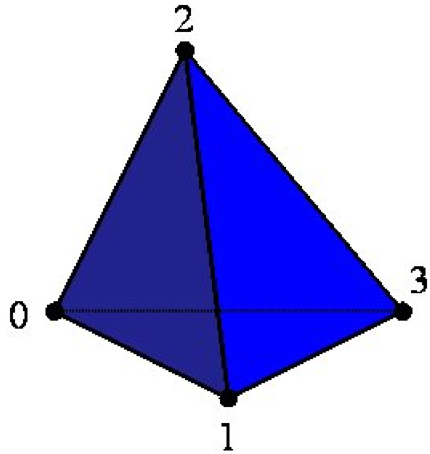
Example:



$$V = \{0, 1, 2, 3, 4, 5, 6\}$$

$$S = \left\{ \begin{array}{l} \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \\ \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \\ \{4,5\}, \{4,6\}, \{5,6\}, \{0,1,2\}, \{4,5,6\} \end{array} \right\}$$

Drawbacks of simplicial complexes.



Example: 2-sphere :

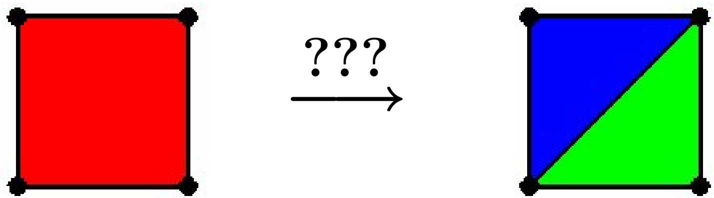
Needs 4 vertices, 6 edges, 4 triangles.

The simplicial set model needs only

1 vertex + 1 "triangle"

but an infinite number of degenerate simplices...

Product?



$\boldsymbol{\Delta^1 \times \Delta^1 = I \times I}$?

In general, constructions are difficult

with simplicial complexes.

Main support for the notion of simplicial set:

The Δ category.

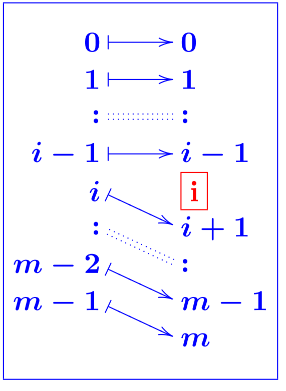
Objects: $\underline{0} = \{0\}$, $\underline{1} = \{0, 1\}$, $\ldots$, $\underline{m} = \{0, 1, \ldots, m\}$, $\ldots$

Morphisms:

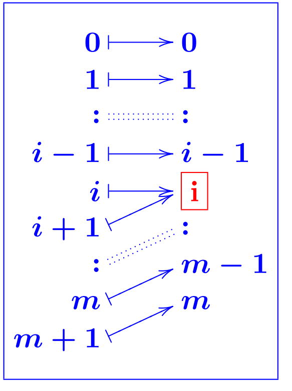
$$\Delta(\underline{m}, \underline{n}) = \{\alpha : \underline{m} \nearrow \underline{n} \;\underline{\mathrm{st}}\; (k \leq \ell \Rightarrow \alpha(k) \leq \alpha(\ell))\}.$$

Examples:

$\partial^m_{\boxed{i}} =$



$\eta^m_{\boxed{i}} =$



Proposition: Any Δ-morphism $\alpha : \underline{m} \nearrow \underline{n}$

has a unique expression:

$$\alpha = \partial_{i_1}^{n} \cdots \partial_{i_{n-p}}^{p+1} \, \eta_{j_{m-p}}^{p} \cdots \eta_{j_1}^{m-1}$$

with $i_1 > \cdots > i_{n-p}$ and $j_{m-p} < \cdots < j_1$ if

$$\alpha : \underline{m} \nearrow \underline{p} \nearrow \underline{n}$$

is the unique Δ-factorisation

through a surjection and an injection.

Example:

$$\Delta(\underline{4}, \underline{5}) \ni \alpha = \boxed{\begin{array}{c} 0 \rightarrow 0 \\ 1 \rightarrow 0 \\ 2 \rightarrow 0 \\ 3 \rightarrow 2 \\ 4 \rightarrow 2 \end{array}} = \partial_5^5 \, \partial_4^4 \, \partial_3^3 \, \partial_1^2 \, \eta_0^1 \, \eta_1^2 \, \eta_3^3$$

Definition: A simplicial set is

a contravariant functor $X : \Delta \to \mathbf{Sets}$.

Example: K simplicial complex $\mapsto X_K$ simplicial set.

$K = (V, S)$ with V totally ordered.

$\mapsto X_K = (\{X_m\}_{\underline{m} \in \mathrm{Ob}(\Delta)}, \{X_\alpha\}_{\alpha \in \mathrm{Mrp}(\Delta)})$ with:

$X_m = \{(v_0 \leq \cdots \leq v_m) \ \underline{\mathrm{st}} \ \{v_0, \ldots, v_m\} \in S\}$;

$\quad = \{\chi : \underline{m} \nearrow V$ monotone and S-compatible$\}$;

$\Delta(\underline{n}, \underline{m}) \ni [\alpha : \underline{n} \nearrow \underline{m}] \mapsto [X_\alpha : X_m \to X_n : \chi \mapsto \chi \circ \alpha]$.

Definition: A simplicial set is

a contravariant functor $X : \Delta \to \mathrm{Sets}$.

Example: K simplicial complex $\longmapsto X_K$ simplicial set.

$K = (V, S)$ with V totally ordered.

$\mapsto X_K = (\{X_m\}_{\underline{m} \in \mathrm{Ob}(\Delta)}, \{X_\alpha\}_{\alpha \in \mathrm{Mrp}(\Delta)})$ with:

$$X_m = \{(v_0 \leq \cdots \leq v_m) \text{ st } \{v_0, \ldots, v_m\} \in S\};$$

$$= \{\chi : \underline{m} \to V \text{ monotone and } S\text{-compatible}\};$$

$$\Delta(\underline{n}, \underline{m}) \ni [\alpha : \underline{n} \to \underline{m}] \mapsto [X_\alpha : X_m \to X_n : \chi \mapsto \chi \circ \alpha].$$

Definition : Let $X = (\{X_m\}, \{X_\alpha\})$ be a simplicial set.

The geometric realization of X is defined by:

$$|X| = (\coprod_m X_m \times \Delta^m)/ \sim$$

with $(x, \alpha_* t) \sim (\alpha^* x, t)$ for $x \in X_m$, $t \in \Delta^n$, $\alpha \in \Delta(\underline{n}, \underline{m})$.

Proposition: Every point of $|X|$ has

a unique representant $(x, t) \in X_m \times \Delta_m$ satisfying:

1. x is a non-degenerate simplex;

2. $t \in \text{Int}\, \Delta^m$.

What is a non-degenerate simplex?

Definition: A simplex $\sigma \in X_m$ is degenerate

if $\sigma = \alpha^* \sigma'$ where $\sigma' \in X_n$ with $n < m$.

Degenerate = Necessary consequence

of another simplex in lower dimension.

Proposition: Every simplex σ has a unique expression

$\sigma = \alpha^* \sigma'$ with σ' non-degenerate and α surjective.

Every degenerate simplex is a degeneracy

of a unique non-degenerate simplex.

Corollary: $|X| = (\coprod_m X_m^{ND} \times \Delta^m) / \sim_{ND}$.

$\Rightarrow$ In fact, only the non-degenerate simplices

are visible in the geometric realization.

Example 1: Standard 3-simplex D (solid tetrehedron).

$$\boldsymbol{m} \in \mathbb{N} \Rightarrow \boldsymbol{D_m} = \boldsymbol{\Delta}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{3}})$$

$$[\boldsymbol{\alpha} \in \boldsymbol{\Delta}(\underline{\boldsymbol{n}}, \underline{\boldsymbol{m}})] + [\boldsymbol{\sigma} \in \boldsymbol{D_m} = \boldsymbol{\Delta}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{3}})]$$
$$\Rightarrow [\boldsymbol{D_\alpha}(\boldsymbol{\sigma}) = \boldsymbol{\alpha}^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \circ \boldsymbol{\alpha} \in \boldsymbol{D_n} = \boldsymbol{\Delta}(\underline{\boldsymbol{n}}, \underline{\boldsymbol{3}})].$$

$$\Rightarrow [\boldsymbol{\sigma} \in \boldsymbol{D_m} = \boldsymbol{\Delta}(\underline{\boldsymbol{m}}, \underline{\boldsymbol{3}}) \text{ non-degenerate}] \Leftrightarrow [\boldsymbol{\sigma} \text{ injective}].$$

$$\Rightarrow \begin{cases} \#D_0^{ND} = 4 \\ \#D_1^{ND} = 6 \\ \#D_2^{ND} = 4 \\ \#D_3^{ND} = 1 \\ \#D_m^{ND} = 0 \text{ if } m \geq 4 \end{cases} \Rightarrow \text{standard model of } \boldsymbol{\Delta}_3.$$

Example 2: $X_m = \{*_m\} \coprod \Delta^{\mathrm{srj}}(\underline{m}, \underline{2})$.

Given $\sigma \in X_m$ and $\alpha \in \Delta(\underline{n}, \underline{m})$,

$$X_\alpha(\sigma) = \alpha^*(\sigma) = \ ?$$

1. $[\sigma = *_m] \Rightarrow [\alpha^*(\sigma) = *_n]$;

2. $\sigma \in \Delta^{\mathrm{srj}}(\underline{m}, \underline{2})$, consider $\sigma \circ \alpha$:

   (a) If $\sigma \circ \alpha \in \Delta^{\mathrm{srj}}(\underline{n}, \underline{2})$, then $\alpha^*(\sigma) = \sigma \circ \alpha$;

   (b) Otherwise $\alpha^*(\sigma) = *_n$.

Exercise: $X_0^{ND} = \{*_0\}$, $X_2^{ND} = \{\mathrm{id}_{\underline{2}}\}$,

others X_m^{ND} are empty.

Corollary: $|X| = S^2$.

Example 3: $G = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{cases} X_m = G^{\underline{m}} = \{\underline{m} \to G\}. \\ [\sigma : \underline{m} \to G] + [\alpha : \underline{n} \to \underline{m}] \Rightarrow [\alpha^*(\sigma) = \sigma \circ \alpha : \underline{n} \to G]. \end{cases}$$

$|X| =$???

Exercise: $\sigma : \underline{m} \to G$ is degenerate

if and only if two consecutive images are equal.

Corollary: $X_m^{ND} = \{(0, 1, 0, 1, \ldots), (1, 0, 1, 0, \ldots)\}$.

$\# X_0^{ND} = 2 \Rightarrow |X|_0 = 2$ points $= S^0$.

$\# X_1^{ND} = 2 \Rightarrow |X|_1 = 2$ edges attached to 2 points $= S^1$.

$\# X_2^{ND} = 2 \Rightarrow |X|_2 = 2$ "triangles" attached to $S^1 = S^2$.

$\cdots \cdots$

$\Rightarrow |X| = \varinjlim S^n = S^\infty$.

Example 4: Let us consider again the previous example X.

A canonical group action $\mathbb{Z}_2 \times X \to X$ is defined.

Dividing X by this action produces a new simplicial set Y.

In particular $\# X_m^{ND} = 2 \Rightarrow \# Y_m^{ND} = 1$.

And $|Y| = S^\infty / \mathbb{Z}_2 = P^\infty \mathbb{R} =$ infinite real projective space.

Example 5: Replacing $\mathbb{Z}_2$ by any discrete group G produces

in the same way the fundamental space $K(G, 1)$.

Definition: $K(G, 1)$ is the base space of a Galoisian covering

where the structural group is G

and the total space is topologically contractible.

Product construction for simplicial sets.

$\boldsymbol{X} = (\{\boldsymbol{X}_m\}, \{\boldsymbol{X}_\alpha\})$, $\boldsymbol{Y} = (\{\boldsymbol{Y}_m\}, \{\boldsymbol{Y}_\alpha\})$ **two simplicial sets.**

$\boldsymbol{Z} = \boldsymbol{X} \times \boldsymbol{Y} =$ **???**

Simple and natural definition:

$\boldsymbol{Z} = \boldsymbol{X} \times \boldsymbol{Y}$ **defined by** $\boldsymbol{Z} = (\{\boldsymbol{Z}_m\}, \{\boldsymbol{Z}_\alpha\})$ **with:**

$\boldsymbol{Z}_m = \boldsymbol{X}_m \times \boldsymbol{Y}_m$

If $\boldsymbol{\Delta}(\underline{\boldsymbol{n}}, \underline{\boldsymbol{m}}) \ni \boldsymbol{\alpha} : \underline{\boldsymbol{n}} \nearrow \underline{\boldsymbol{m}}$**:**

$$\boldsymbol{Z}_\alpha \colon \boldsymbol{X}_m \times \boldsymbol{Y}_m \xrightarrow{X\alpha \times Y_\alpha} \boldsymbol{X}_n \times \boldsymbol{Y}_n$$

Example: $\boldsymbol{I \times I = \Delta^1 \times \Delta^1 =}$???

$\boldsymbol{Z_m = \Delta(\underline{m}, \underline{1}) \times \Delta(\underline{m}, \underline{1})}$

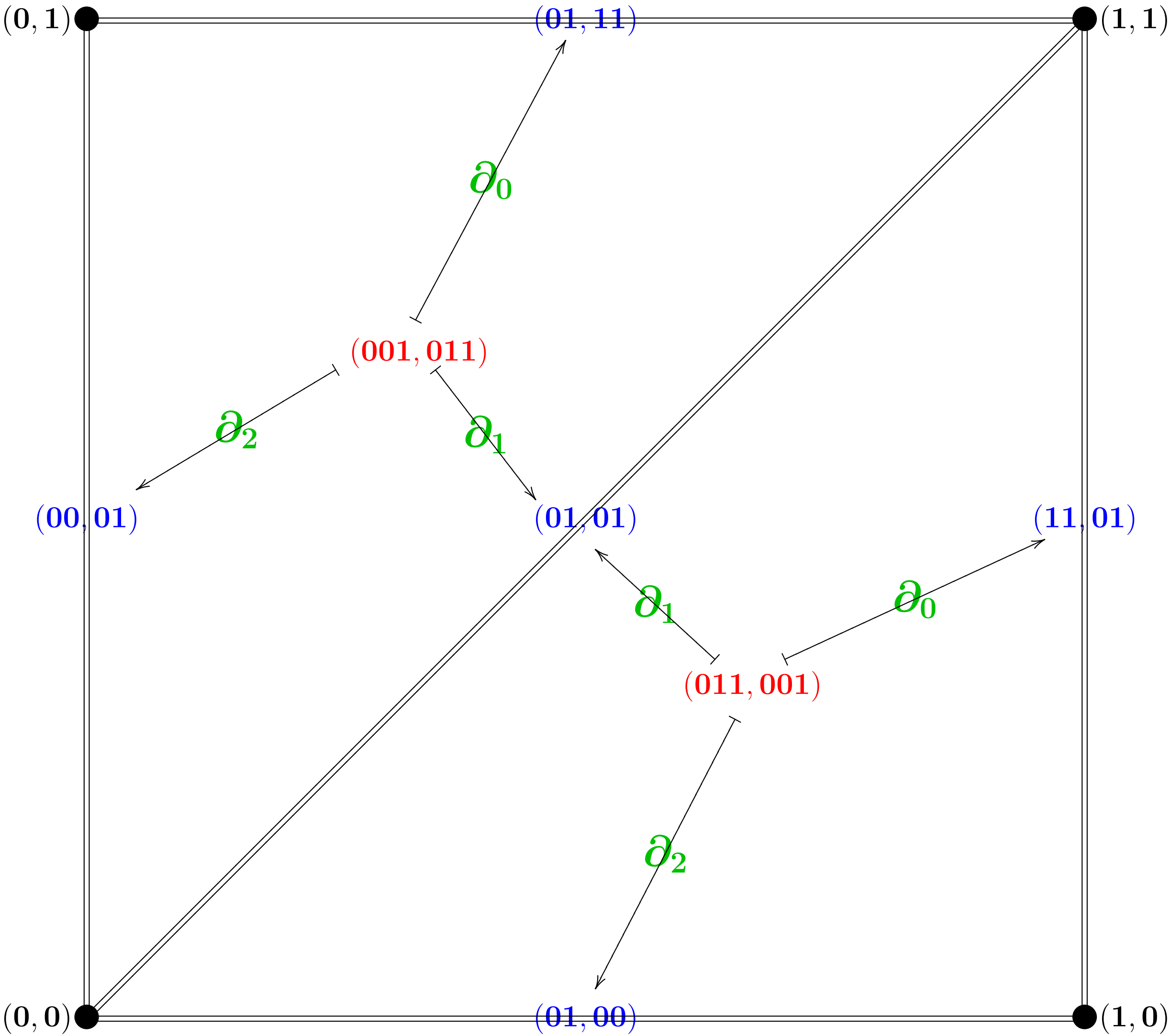
$\mathbf{(011, 001) \in \boldsymbol{Z_2}}$

$\boldsymbol{\partial_0 : 01 \nearrow 012} \qquad \boldsymbol{\partial_0(011, 001) = (11, 01)}$

$\boldsymbol{\partial_1 : 01 \nearrow 012} \qquad \boldsymbol{\partial_1(011, 001) = (01, 01)}$

$\boldsymbol{\partial_2 : 01 \nearrow 012} \qquad \boldsymbol{\partial_2(011, 001) = (01, 00)}$

Picture ???


(0,1)
(01,11)
(1,1)
∂_0
(001,011)
∂_2
∂_1
(00,01)
(01,01)
(11,01)
∂_1
∂_0
(011,001)
∂_2
(0,0)
(01,00)
(1,0)

Twisted Products.

$B = (\{B_m\}, \{B_\alpha\})$ with:

$$\begin{cases} B_0^{ND} = \{*\} \\ B_1^{ND} = \{s_1\} \\ B_m^{ND} = \emptyset \text{ if } m \geq 2 \end{cases} \Rightarrow B = \text{ standard model of } S^1.$$

$G = (\{G_m\}, \{G_\alpha\})$

with $G_m = \mathbb{Z}$ $(\forall m \in \mathbb{N})$ and $G_\alpha = \mathrm{id}_{\mathbb{Z}}$ $(\forall \alpha \in \Delta(\underline{m}, \underline{n}))$.

Exercise: $|G| = \mathbb{Z}$.

$\Rightarrow B \times G = S^1 \times \mathbb{Z} =$ stack of circles.

In particular: $\partial_0(s_1, k_1) = (*, k_0)$.

Definition: $\boxed{B = \text{simplicial set}} + \boxed{G = \text{simplicial group}}$.

A twisting function $\tau : B \to G$

is a family $(\tau_m : B_m \to G_{m-1})_{m \geq 1}$ satisfying:

$\partial_i \tau(b) \;=\; \tau\partial_{i+1}(b) \;\; (i \geq 1)$	$\partial_0 \tau(b) \;=\; (\tau\partial_0(b))^{-1}\tau\partial_1(b)$
$\eta_i \tau(b) \;=\; \tau\eta_{i+1}(b) \;\; (i \geq 0))$	$\tau\eta_0(b) \;=\; e_{m-1} \;\; (b \in B_m)$

New face operators for $B \times_\tau G$:

$\partial_i(b, g) = (\partial_i b, \partial_i g) \;\; (i \geq 1)$	$\partial_0(b, g) = (\partial_0 b, \tau(b).\partial_0(b))$

Example 1: $B = S^1$, $G = \mathbb{Z}$, $\tau_1(s_1) = 0_0 \in G_0$

$$\Rightarrow B \times_\tau G = S^1 \times \mathbb{Z} = \text{trivial product}.$$

In particular $\partial_0(s_1, \tau(s_1).k_1) = (*, k_0)$.

Example 2: $B = S^1$, $G = \mathbb{Z}$, ${\tau'}_1(s_1) = 1_0 \in G_0$

$$\Rightarrow B \times_\tau G = S^1 \times \mathbb{Z} = \text{twisted product}.$$

Now $\partial_0(s_1, k_1) = (*, \tau'(s_1).k_0) = (*, (k+1)_0)$.

Daniel Kan's fantastic work ($\sim$ 1960 – 1980).

Every "standard" natural topological construction process has a translation in the simplicial world.

Frequently the translation is even "better".

Typical example. The loop space construction in ordinary topology gives only an *H*-space (= group up to homotopy).

Kan's loop space construction produces a genuine simplicial group, playing an essential role in Algebraic Topology.

Conclusion: Simplicial world = Paradise! ???

Simplicial group model for the unit circle $S^1 \subset \mathbb{C}^2$?

$$S^1 = X = (\{X_m\}, \{X_\alpha\}) = ???$$

Eilenberg-MacLane solution (1955):

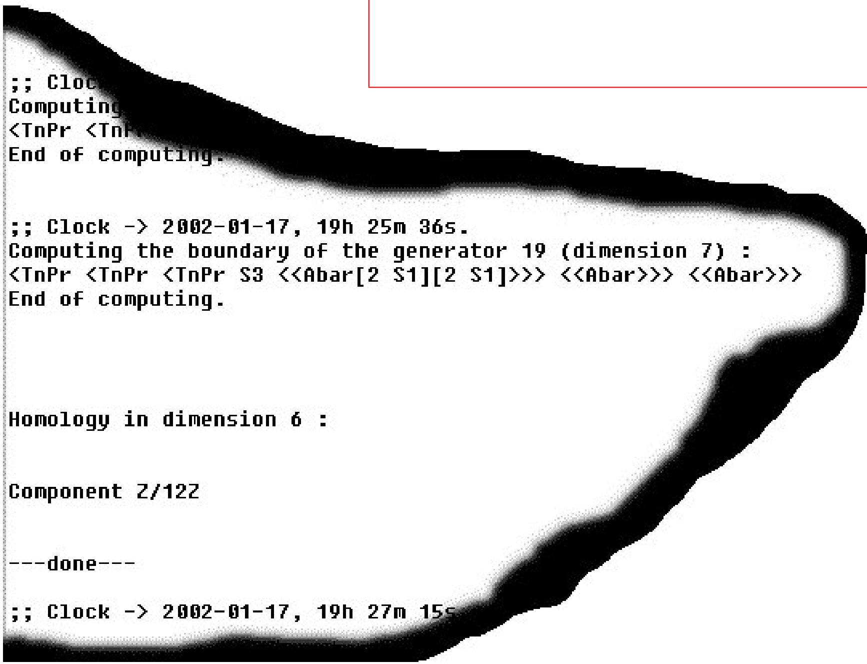
$$X_m = \mathbb{Z}^m + \text{ appropriate } X_\alpha\text{'s}$$

$\Rightarrow$ X highly infinite in every dimension.

Kan (1960) $\Rightarrow$ Eilenberg-MacLane model is the minimal one!

$\Rightarrow$ Simplicial world = Paradise or Hell???

The END



Francis Sergeraert, Institut Fourier, Grenoble, France
Castro-Urdiales, January 9-13, 2006