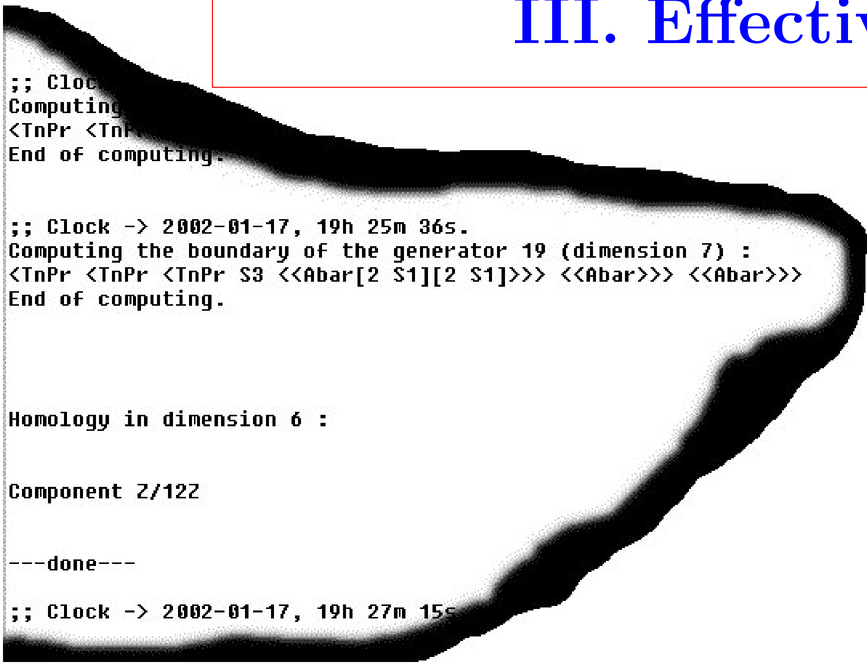
Algebraic Topology

(Castro-Urdiales tutorial)

III. Effective Homology



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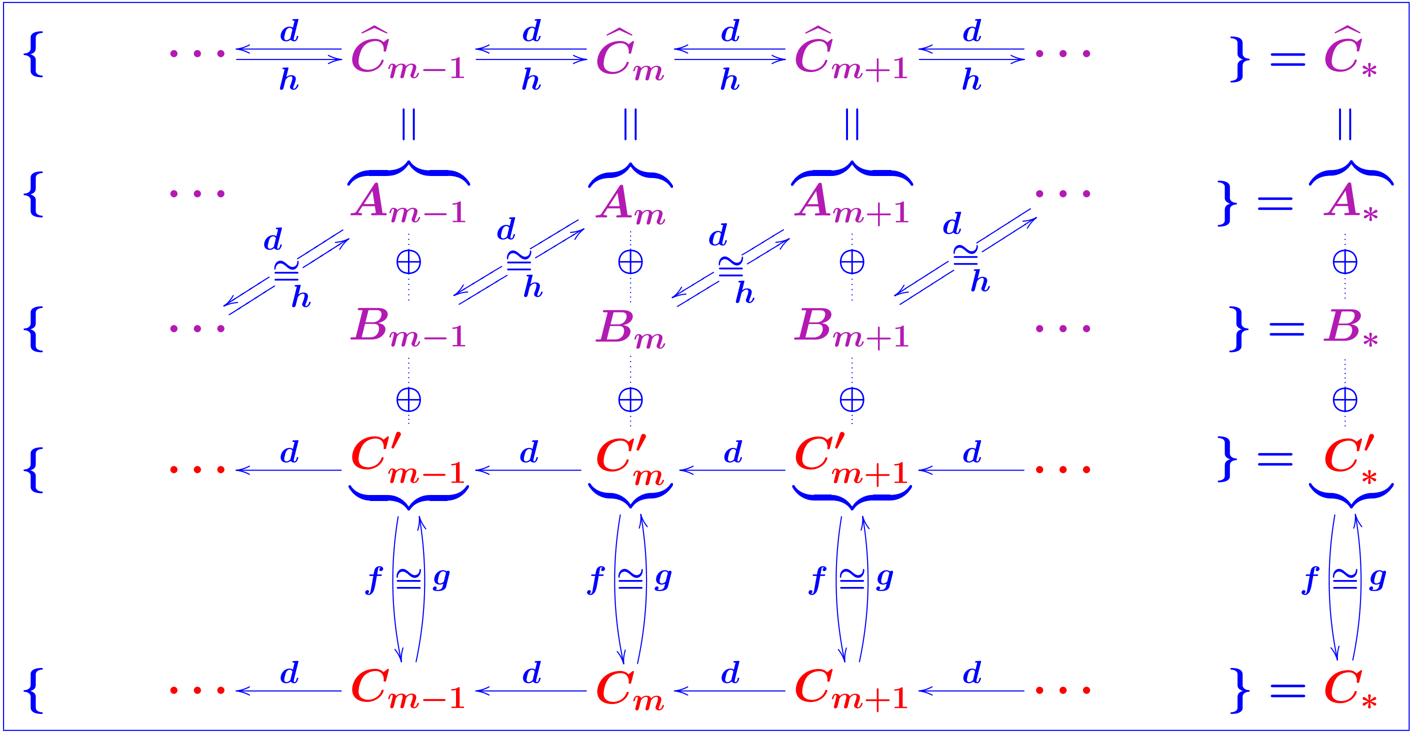
<u>Definition</u>: A (Homological) reduction is a diagram:

$$\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ **and** C_* $=$ **(free $\mathbb{Z}$-) chain complexes.**
2. f **and** g $=$ **chain complex morphisms.**
3. h $=$ **homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ **and** $hh = 0$.



$$A_* = \ker f \cap \ker h \qquad B_* = \ker f \cap \ker d \qquad C'_* = \operatorname{im} g$$

$$\widehat{C}_* = A_* \oplus B_* \text{ exact} \oplus C'_* \cong C_*$$

.

Let ρ : $h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftleftarrows}} C_*$ be a reduction.

Frequently:

1. **$\widehat{C}_*$ is a locally effective chain complex:**
   its homology groups are unreachable.

2. **C is an effective chain complex:**
   its homology groups are computable.

3. **The reduction ρ is an entire description of the homological nature of $\widehat{C}_*$.**

4. **Any homological problem in $\widehat{C}_*$ is solvable**
   thanks to the information provided by ρ.

$$\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

1. What is $H_n(\widehat{C}_*)$? Solution: Compute $H_n(C_*)$.

2. Let $x \in \widehat{C}_n$. Is x a cycle? Solution: Compute $d_{\widehat{C}_*}(x)$.

3. Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. Let $x, x' \in \widehat{C}_n$ be homologous cycles.

   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.

<u>Definition</u>: (C_*, d) = given chain complex.

A perturbation δ: $C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

<u>Problem</u>: Let ρ : $\boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

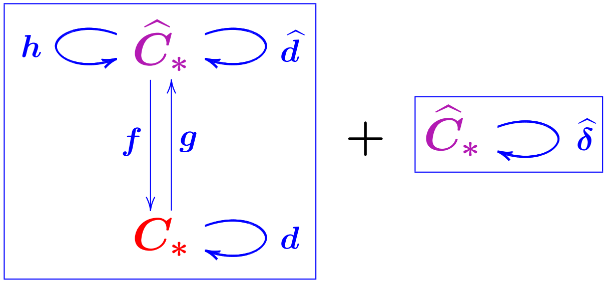
$$? : \boxed{? \circlearrowright (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \underset{?}{\overset{?}{\leftrightarrows}} (C_*, ?)}$$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma":

Given:

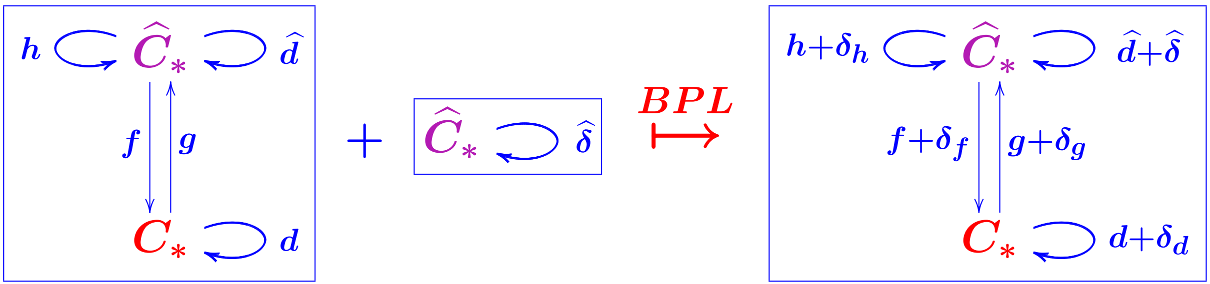


satisfying:

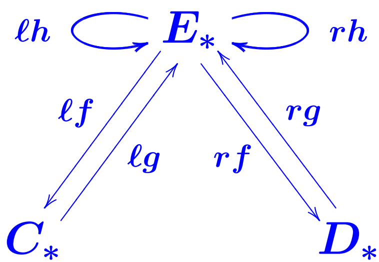
1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$;

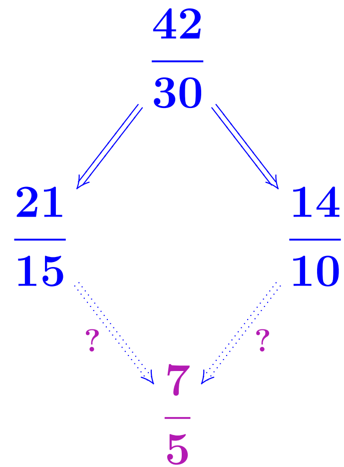
2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:



<u>**Definition**</u>**: A (strong chain-) equivalence $\varepsilon : C_* \Longleftrightarrow D_*$ is a pair of reductions $C_* \overset{\ell\rho}{\Lleftarrow} E_* \overset{r\rho}{\Rrightarrow} D_*$:**





Normal form problem ??

More structure often necessary in C_*.

Definition: An object with effective homology $\mathcal{X}$ is a 4-tuple:

$$\mathcal{X} = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, …)
2. $C_*(X)$ = the chain complex "traditionally" associated to X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \overset{\varepsilon}{\Lleftarrow\!\!\Rrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_{1*}, \ldots, X_{n*}$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_{1*}, \ldots, X_{n*}) \mapsto X_*.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}: (\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n})$$

$$\mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is $\boxed{\text{perfectly stable}}$

and can be again used with X for further calculations.

Detailed study of a particular case.

Constructive version of Serre spectral sequence.

Theorem: There exists an algorithm:

$$\mathbf{TwPr}_{EH} : \boxed{\begin{array}{c} \boxed{F, C_*(F), EC_*^F, \varepsilon_F} \\ \tau : B_* \to F_{*-1} \\ \boxed{B, C_*(B), EC_*^B, \varepsilon_B} \end{array}} \mapsto \boxed{E, C_*(E), EC_*^E, \varepsilon_E}$$

with:

1. B, F = **simplicial sets with effective homology.**
2. $\tau : B_* \to F_{*-1}$ = **twisting function.**
3. $\boxed{F \hookrightarrow \boxed{E = B \times_\tau F} \to B}$ = **fibration = twisted product.**

Composition of equivalences:

$$\boxed{\begin{array}{c} \boxed{A_* \Lleftarrow B_* \Rrightarrow C_*} \\ + \\ \boxed{C_* \Lleftarrow D_* \Rrightarrow E_*} \end{array} \longmapsto \boxed{A_* \Lleftarrow F_* \Rrightarrow E_*}}$$

with F_* an appropriate by-product of the diagram:

$$B_* \Rrightarrow C_* \Lleftarrow D_*$$

Tensor product of equivalences:

$$\boxed{\begin{array}{c} \boxed{A_* \Lleftarrow B_* \Rrightarrow C_*} \\ + \\ \boxed{A'_* \Lleftarrow B'_* \Rrightarrow C'_*} \end{array}} \longmapsto \boxed{A_* \otimes A'_* \Lleftarrow B_* \otimes B'_* \Rrightarrow C_* \otimes C'_*}$$

Particular case of the trivial product: $E = B \times H$.

<u>Theorem</u> (Eilenberg-Zilber): A and B given simplicial sets.

There exists a canonical reduction:

$$\rho_{EZ} : C_*(A \times B) \Rrightarrow C_*(A) \otimes C_*(B).$$

Dimensions in the particular case: $A = B = \Delta^7$.

n	$\times \Rrightarrow \otimes$		n	$\times \Rrightarrow \otimes$		n	$\times \Rrightarrow \otimes$	
0	64	64	5	759,752	11,424	10	1,475,208	1,820
1	1,232	448	6	1,549,936	12,868	11	673,134	560
2	11,872	1,680	7	2,360,501	11,440	12	208,824	120
3	69,524	4,256	8	2,703,512	8,008	13	39,468	16
4	272,944	9,527	9	2,322,180	4,368	14	3,432	1

<u>Corollary</u>: A and B = given simplicial sets with effective homology.

Then the trivial product $A \times B$ is a simplicial set with effective homology.

<u>Proof</u>:

$$C_*(A \times B) \overset{\text{id}}{\Lleftarrow\!\!\Lleftarrow} C_*(A \times B) \overset{\rho_{EZ}}{\Rrightarrow} C_*(A) \otimes C_*(B) \overset{\varepsilon_A \otimes \varepsilon_B}{\Lleftarrow\!\!\Longleftrightarrow\!\!\Rrightarrow} EC_*^A \otimes EC_*^B$$

with $EC_*^A \otimes EC_*^B$ effective.

Composition of equivalences $\Rightarrow$

$$C_*(A \times B) \overset{\varepsilon_{A \times B}}{\Lleftarrow\!\!\Longleftrightarrow\!\!\Rrightarrow} EC_*^A \otimes EC_*^B$$

QED

General case: $F \hookrightarrow B \times_\tau F \to B$.

$\boxed{B, C_*(B), EC_*^B, \varepsilon_B} + \boxed{F, C_*(F), EC_*^F, \varepsilon_F} + \boxed{\tau : B_* \to F_{*-1}}$

Theorem (**Easy Basic Perturbation Lemma**):

$\boxed{\rho : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)} + \boxed{\delta : C_* \to C_{*-1} = \text{perturbation of } d}$

$\mapsto \boxed{\rho' : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)}$.

Proof: $(\widehat{C}_*, \widehat{d}) = (A_*, \widehat{d}) \oplus (C'_*, d')$ **with** $(C'_*, d') \cong (C, d)$.

Copy into (C'_*, d') **the perturbation** $\delta \quad \mapsto (C'_*, d' + \delta')$.

Solution $= \rho : ((A_*, \widehat{d}) \oplus (C'_*, d' + \delta')) \Rrightarrow (C_*, d + \delta)$.

QED

Constructing the effective homology of $C_*(B \times_\tau F)$.

Initial diagram.

$$C_*(B \times F) \Rrightarrow C_*(B) \otimes C_*(F) \Lleftarrow \widehat{C}_*^B \otimes \widehat{C}_*^F \Rrightarrow EC_*^B \otimes EC_*^F$$

Difficult BPL (= Ed. Brown Theorem) ⇒

$$C_*(B \times_\tau F) \Rrightarrow C_*(B) \otimes_t C_*(F) \,|\, \Lleftarrow \widehat{C}_*^B \otimes \widehat{C}_*^F \Rrightarrow EC_*^B \otimes EC_*^F$$

Easy BPL ⇒

$$C_*(B \times_\tau F) \Rrightarrow C_*(B) \otimes_t C_*(F) \Lleftarrow \widehat{C}_*^B \otimes_t \widehat{C}_*^F \,|\, \Rrightarrow EC_*^B \otimes EC_*^F$$

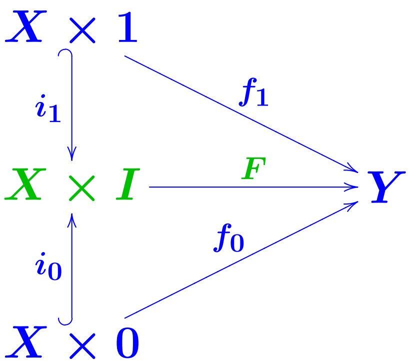
Difficult BPL ⇒

$$C_*(B \times_\tau F) \Rrightarrow C_*(B) \otimes_t C_*(F) \Lleftarrow \widehat{C}_*^B \otimes_t \widehat{C}_*^F \Rrightarrow EC_*^B \otimes_t EC_*^F$$

QED

Important hypothesis missing in the previous slides!!

<u>Definition</u>: Two continuous maps $f_0, f_1 : X \to Y$ are homotopic if $\exists F : X \times I \to Y$ such that $F|X \times 0 = f_0$ and $F|X \times 1 = f_1$.



Definition: A connected space X is simply connected if two arbitrary continuous maps $f_0, f_1 : S^1 \to X$ are homotopic.

Examples: $\mathbb{R}^2$ is simply connected, $\mathbb{R}^2 - \{(0,0)\}$ is not.

S^2 is simply connected, $P^2(\mathbb{R}) = S^2/\mathbb{Z}_2$ is not.

Theorem: There exists an algorithm:

$$\mathrm{TwPr}_{EH} : \boxed{\begin{array}{c} \boxed{F, C_*(F), EC_*^F, \varepsilon_F} \\ \tau : B_* \to F_{*-1} \\ \boxed{B, C_*(B), EC_*^B, \varepsilon_B} \end{array}} \mapsto \boxed{E, C_*(E), EC_*^E, \varepsilon_E}$$

with $E = B \times_\tau F$

on condition that the base space B is simply connected.

Poincaré group $= \pi_1(X)$:

measures the lack of simple connectivity of X.

<u>Theorem</u> (Novikov-Rabin) : There does not exist

any decision algorithm for the problem:

$G =$ finitely presented group; $G = 0$???

$K =$ finite simplicial complex $\Rightarrow \pi_1(X)$ finitely presented.

$G =$ arbitrary finitely presented group; $\exists K$ <u>st</u> $\pi_1(K) = G$.

<u>Corollary</u>: No decision algorithm for the problem:

$K =$ finite simplicial complex; is K simply connected ???

Theorem (+ Gödel-Turing): Let $(K_n)_{n\in\mathbb{N}}$ be the sequence of finite simplicial complexes. There exists n_0 such that:

1. $\forall n < n_0$ the problem $<\pi_1(K_n) \overset{???}{=} 0>$ is decidable.
2. $\pi_1(K_{n_0}) \neq 0$, but there does not exist a proof.
3. The knowledge of this n_0 is unreachable.

Strong differences between "Simply Connected Topology" and "General Topology".

Definition: A homotopy equivalence $X \underset{f}{\overset{g}{\leftrightarrows}} Y$ is a pair of continuous maps such that fg and gf are homotopic to the identity.

Definition: A contractible space is equivalent to a point.

Definition: A right inverse of a topological space X is some topological group R_X and a twisting function $\tau : X \longrightarrow R_X$ such that $X \times_{\tau_X} R_X$ is contractible.

Definition: A left inverse of a topological group G is some topological space L_G and some twisting function $\tau : L_G \longrightarrow$ such that $L_G \times_{\tau_G} G$ is contractible.

Theorem: These inverses are well defined up to homotopy.

"Theorem": Some spectral sequences (Eilenberg-Moore) "compute" the homology groups of these inverses.

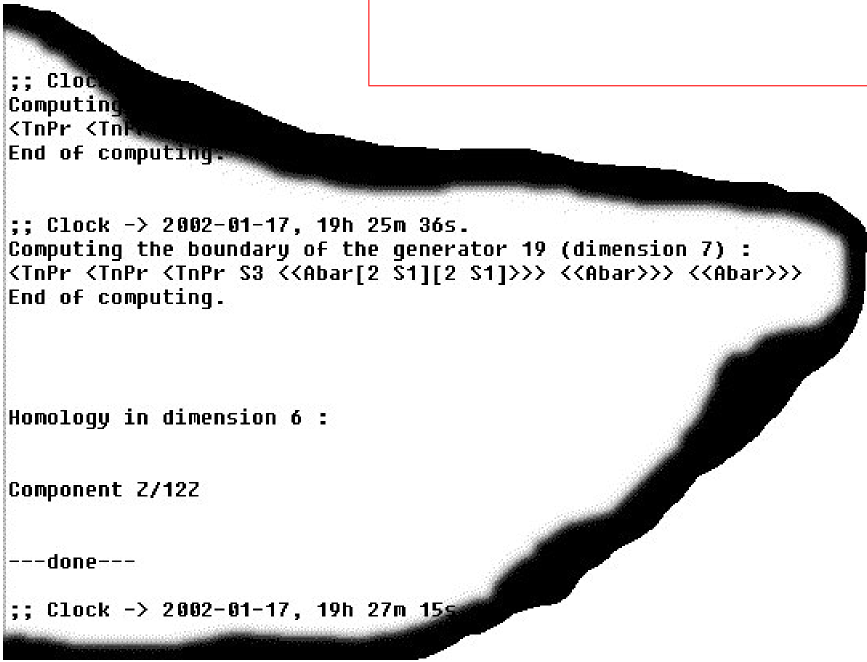
Theorem: There exist algorithms:

$$R_{EH} : X_{EH} \mapsto (R_X)_{EH} \qquad\qquad L_{EH} : G_{EH} \mapsto (L_G)_{EH}$$

working when X is simply connected and G connected.

When inversion is available, division is available as well...

The END



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