The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

1958-16

Summer School and Conference Mathematics, Algorithms and Proofs

11 - 29 August 2008

Point-Free Topology

Erik Palmgren
Department of Mathematics, Uppsala University, Sweden

Point-Free Topology[1]

Erik Palmgren
Uppsala University, Dept. of Mathematics

MAP Summer School 2008,
International Centre for Theoretical Physics.
Trieste, August 11 – 22, 2008.

[1] Notes and slides will materialise at **www.math.uu.se/˜palmgren**

Topology and choice principles

It is well-known that some basic theorems classical topology use (and require) the full Axiom of Choice:

Tychonov's Theorem (AC): If $(X_i)_{i \in I}$ is a family of (covering) compact spaces then the product topology

$$\prod_{i \in I} X_i$$

is compact.

Special case: The *Cantor space* $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ is compact.

Constructive topology

To carry out topology on a constructive foundation it is necessary to come to grips with compactness.

Brouwer's solution: The compactness of the Cantor space follows from the nature of choice sequences. This the **Fan Theorem**. Moreover, it implies that the interval $[0, 1]$ is compact.

Kleene: The Cantor space is not compact under a recursive realizability interpretation.

E. Bishop: Restricting to metric spaces, covering compactness should be replaced by total boundedness (*Foundations of Constructive Analysis*, 1967).

Point-free topology

A quite early idea in topology: study spaces in terms of the relation between the open sets. (Wallman 1938, Menger 1940, McKinsey & Tarski 1944, Ehresmann, Benabou, Papert, Isbell,...)

For a topological space X the frame of open sets $(\mathcal{O}(X), \subseteq)$ is a complete lattice satisfying an infinite distributive law

$$U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I} U \wedge V_i$$

(or, equivalently, is a complete Heyting algebra). The inverse mapping of a continuous function $f : X \longrightarrow Y$ gives rise to a lattice morphism

$$f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$

which preserves arbitrary suprema $\bigvee$.

Many important properties of topological spaces may be expressed without referring to the points, but only referring to the relation between the open sets. For example a topological space X is *compact* if for any family of open sets U_i $(i \in I)$ in X we have

if $\bigcup_{i \in I} U_i = X$, then there are $i_1, \ldots, i_n \in I$ so that $U_{i_1} \cup \cdots \cup U_{i_n} = X$

That is, every open cover of the space has a finitely enumerable (f.e.) subcover.

Recall: A *lattice* is a partially ordered set $(A, \leq)$ where every finite list $a_1, \ldots, a_n$ of elements has a supremum

$$a_1 \vee \cdots \vee a_n = \bigvee_{i \in \{1,\ldots,n\}} a_i$$

and an infimum

$$a_1 \wedge \cdots \wedge a_n = \bigwedge_{i \in \{1,\ldots,n\}} a_i$$

For $n = 0$ these are respectively $\perp$ (the smallest element) and T (the largest element).

A lattice $(A, \leq)$ is a *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

(Exercise show that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ follows.)

A lattice A is *complete* if $\sup S$ exists for every subset $S \subseteq A$.

Proposition: In a complete lattice A the infimum of $S \subseteq A$, is given by

$$\bigwedge S = \bigvee \{x \in A : x \text{ is a lower bound of } S\}.$$

Def A *frame* (or *locale*) is a complete lattice A which satisfies the *infinite distributive law:*

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} a \wedge b_i$$

for any subset $\{b_i : i \in I\} \subseteq A$.

Example

If X is a topological space, then its open sets $(\mathcal{O}(X), \subseteq)$ ordered by inclusion form a frame. Here

$$U \wedge V = U \cap V \qquad \bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i.$$

However,

$$\bigwedge_{i \in I} U_i = \mathrm{Interior}(\bigcap_{i \in I} U_i).$$

A lattice A is a *Heyting-algebra* if there is a binary operation $(\cdot \Rightarrow \cdot) : A \times A \longrightarrow A$ so that for all $a, b, c \in A$:

$$a \wedge b \leq c \Longrightarrow a \leq (b \Rightarrow c).$$

(Cf. laws for implication.)

Theorem. Let A be a complete lattice. Then A is Heyting-algebra iff A is a frame.

To prove the if-direction define

$$(b \Rightarrow c) = \bigvee \{a \in A : a \wedge b \leq c\}.$$

In a Heyting algebra we can define a pseudo-complement

$$\neg a = (a \Rightarrow \bot).$$

We have

$$a \wedge \neg a = \bot.$$

However $a \vee \neg a = \top$ is in general false:

Example

Let $\mathbb{R}$ be the real line with the usual topology. Then for an open subset $U \subseteq \mathbb{R}$ we have

$$\neg U = \bigcup \{V \in \mathcal{O}(\mathbb{R}) : V \cap U = \emptyset\}.$$

In particular, $\neg(0,1) = (-\infty, 0) \cup (1, \infty)$ so

$$(0,1) \vee \neg(0,1) \neq \mathbb{R} = \top.$$

A frame morphism $h : A \longrightarrow B$ from between frames is a lattice morphism that preserves infinite suprema:

(i) $h(\top) = \top$,

(ii) $h(a \wedge b) = h(a) \wedge h(b)$,

(iii) $h(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} h(a_i)$.

The frames and frame morphisms form a category, **Frm**.

Example

A typical frame morphism is the pre-image operation $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, where $f : X \longrightarrow Y$ is any continuous function.

For conceptual reasons one considers the opposite category of the category of frames:

Def Let A and B be two locales. A *locale morphism* $f : A \longrightarrow B$ is a frame morphism $f^* : B \longrightarrow A$.

The composition $g \circ f$ of locale morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is given by

$$(g \circ f)^* = f^* \circ g^*.$$

Denote the category of locales and locale morphisms by **Loc**.

The locale (frame) $(\mathcal{O}(\{\star\}), \subseteq)$ that comes from the one point space $\{\star\}$ is denoted $\mathbf{1}$. It is the terminal object of **Loc**. Here $\top = \{\star\}$ and $\bot = \emptyset$.

Since the points of an ordinary topological space X are in 1-1 correspondence with the maps $\{\star\} \longrightarrow X$ (necessarily continuous) we may define what a point of locale is by analogy:

Def. Let A be a locale. A *point of A* is any locale morphism $p : \mathbf{1} \longrightarrow A$.

Note that such point is a frame morphism $p^* : A \longrightarrow \mathbf{1}$, which is completely determined by the elements

$$F_p = \{a \in A : p^*(a) = \mathrm{T}\}$$

sent to T. Thus we may identify points with completely prime filters F on A, i.e. $F \subseteq A$ containing T such that

- $a \in F,\ a \leq b \Longrightarrow b \in F$,
- $a, b \in F \Longrightarrow a \wedge b \in F$,
- $\bigvee_{i \in I} a_i \in F \Longrightarrow (\exists i \in I)\ a_i \in F$.

Fundamental adjunction

Theorem The functor $\Omega : \mathbf{Top} \longrightarrow \mathbf{Loc}$ is left adjoint to the functor $\mathrm{Pt} : \mathbf{Loc} \longrightarrow \mathbf{Top}$, that is, there is an bijection

$$\theta_{X,A} : \mathbf{Loc}(\Omega(X), A) \cong \mathbf{Top}(X, \mathrm{Pt}(A)),$$

natural in X and A.

Here $\Omega(X) = (\mathcal{O}(X), \subseteq)$ and $\Omega(f)^* = f^{-1} : \Omega(Y) \longrightarrow \Omega(X)$ for $f : X \longrightarrow Y$.

$\mathrm{Pt}(A)$ is the set of points of the locale A, and $\mathrm{Pt}(f)(x) = f \circ x$.

A locale is *spatial* if it has enough points to distinguish elements of the locale i.e.

$$a^* = b^* \Longrightarrow a = b.$$

A space X is *sober* if every irreducible closed set is the closure of a unique point. (A nonempty closed C is *irreducible*, if for any closed C', C'': $C \subseteq C' \cup C''$ implies $C \subseteq C'$ or $C \subseteq C''$.)

The adjunction induces an equivalence between the category of sober spaces and category of spatial locales:

$$\mathbf{Sob} \simeq \mathbf{Spa}$$

Remark Ω gives a full and faithful embedding of all Hausdorff spaces into locales (in fact of all sober spaces).

Locale Theory — Formal Topology

Standard Locale Theory can be dealt with in an impredicative constructive setting, e.g. in a topos (Joyal and Tierney 1984).

The predicativity requirements of BISH requires a good representation of locales, formal topologies. They were introduced by **Martin-Löf and Sambin** (Sambin 1987) for this purpose.

$(A, \leq, \lhd)$ is a formal topology if $(A, \leq)$ is a preorder, and $\lhd$ is an abstract cover relation which extends $\leq$. It represents a locale $(\mathrm{Sat}(A), \subseteq)$ whose elements are the saturated subsets $U \subseteq A$, i.e.

$$a \lhd U \Longrightarrow a \in U$$

(A formal topology corresponds to Grothendieck's notion of a site on a preordered set.)

More in detail: A *formal topology* X is a pre-ordered set $(X, \leq)$ of so-called *basic neighbourhoods*. This is equipped with a *covering relation* $a \lhd U$ between elements a of X and subsets $U \subseteq X$ and so that $\{a \in X : a \lhd U\}$ is a subset of X. The cover relation is supposed to satisfy the following conditions:

(Ext) If $a \leq b$, then $a \lhd \{b\}$,

(Refl) If $a \in U$, then $a \lhd U$,

(Trans) If $a \lhd U$ and $U \lhd V$, then $a \lhd V$,

(Loc) If $a \lhd U$ and $a \lhd V$, then $a \lhd U \wedge V$.

Here $U \lhd V$ is an abbreviation for $(\forall x \in U)\, x \lhd V$. Moreover $U \wedge V$ is short for the *formal intersection* $U_{\leq} \cap V_{\leq}$, where $W_{\leq} = \{x \in X : (\exists y \in W) x \leq y\}$, the *down set of W*.

A *point* of a formal topology $X = (X, \leq, \lhd)$ is a subset $\alpha \subseteq X$ such that

(i) α is inhabited,

(ii) if $a, b \in \alpha$, then for some $c \in \alpha$ with $c \leq a$ and $c \leq b$,

(iii) if $a \in \alpha$ and $a \leq b$, then $b \in \alpha$,

(iv) if $a \in \alpha$ and $a \lhd U$, then $b \in \alpha$ for some $b \in U$.

The collection of points in X is denoted $\mathrm{Pt}(X)$. It has a point-set topology given by the open neighbourhoods:

$$a^* = \{\alpha \in \mathrm{Pt}(X) : a \in \alpha\}.$$

We shall often consider different cover relations $\lhd$, $\lhd'$, $\lhd''$, $\ldots$ on one and the same underlying pre-ordered set $(X, \leq)$. There is an natural ordering of cover relations

$$\lhd \subseteq \lhd'$$

which holds iff for all $a \in X$ and for all subsets U of X, $a \lhd U$ implies $a \lhd' U$. We say that $\lhd$ is smaller than $\lhd'$. On a given preorder $(X, \leq)$ there is a smallest cover relation given by

$$a \lhd_s U \Longleftrightarrow_{\mathrm{def}} a \in U_{\leq}.$$

(Check this as an exercise.) If $\leq$ is $=$ on X, this correspond to $(X, \leq, \lhd_s)$ being the discrete topology. There is also a largest cover relation

$$a \lhd_t U \Longleftrightarrow_{\mathrm{def}} \text{ true},$$

which gives the trivial topology.

Many properties of the formal topology X can now be defined in terms of the cover directly. We say that X is *compact* if for any subset $U \subseteq X$

$$X \lhd U \Longrightarrow (\exists \text{ f.e. } U_0 \subseteq U)\ X \lhd U_0.$$

For $U \subseteq X$ define

$$U^\perp = \{x \in X : \{x\} \wedge U \lhd \emptyset\},$$

the *open complement of U*. It is easily checked that $U^\perp \wedge U \lhd \emptyset$, and if $V \wedge U \lhd \emptyset$, then $V \lhd U^\perp$.

A basic neighbourhood a is *well inside* another neighbourhood b if $X \lhd \{a\}^\perp \cup \{b\}$. In this case we write $a \lll b$. A formal topology X is *regular* if its cover relation satisfies

$$a \lhd \{b \in X : b \lll a\}.$$

(Compact Hausdorff = Compact regular.)

For motivation we start with the extended example of the Cantor space.

Let S be a set. Denote by $S^{<\omega}$ the set of finite sequence of elements in S: $\langle a_1, \ldots, a_n \rangle$. Concatenation of sequences u, v is denoted $u \frown v$. Order the sequences by saying that $w \leq u$ iff $w = u \frown v$ for some v.

Denote by S^ω the set of infinite sequences of elements in S, i.e. of functions $\mathbb{N} \longrightarrow S$. We define a topology on S^ω by declaring the basic open sets to be

$$B_{\langle a_1,\ldots,a_n \rangle} = \{x \in S^\omega : (\forall i = 1, \ldots, n)\, x(i-1) = a_i\}.$$

We have $B_{\langle\rangle} = S^\omega$.

By letting $S = \{0,1\}$ we get the so-called *Cantor space* $\mathcal{C} = \{0,1\}^\omega$ of infinite binary sequences. By letting $S = \mathbb{N}$ we obtain the *Baire space* $\mathbb{N}^\omega$.

Classically we can show that S^ω is compact iff S is finite. This uses König's lemma.

Brouwer's Fan Theorem implies that the Cantor space, and in fact every subspace of the form B_v, is covering compact: For any subset $M \subseteq \{0,1\}^{<\omega}$:

$$\bigcup_{u \in M} B_u = B_v \Longrightarrow (\exists \text{ f.e. } F \subseteq M) \bigcup_{u \in F} B_u = B_v.$$

However this "theorem" is really an axiom as it is false under many natural constructive interpretations, for instance recursive realizability interpretations.

For a finite $F = \{u_1, \ldots, u_m\} \subseteq \{0,1\}^{<\omega}$ the covering relation $\bigcup_{u \in F} B_u = B_v$ can be checked by considering sequences of length $n = \max(|u_1|, \ldots, |u_m|)$. We have

$$\bigcup_{u \in F} B_u = B_v \Longleftrightarrow (\forall w \in \{0,1\}^{<\omega})(|w| = n, w \leq v \Rightarrow w \in F_{\leq}).$$

We abbreviate the righthand side as the property $K(v, n, F)$. This property is clearly decidable given v, n and F.

We shall now define a formal topology corresponding to the Cantor space. Let $C = (\{0,1\}^{<\omega}, \leq, \lhd)$ where $\leq$ is as above and $\lhd$ is given by

$$v \lhd U \Longleftrightarrow_{\mathrm{def}} (\exists \text{ f.e. } F \subseteq U)(\exists n)\, K(v, n, F).$$

Now compactness is built-in to the cover relation.

Thm. C is a compact formal topology and $\mathrm{Pt}(C) \cong \mathcal{C}$.

Note: for any u

$$u \lhd \{u \frown \langle 0 \rangle, u \frown \langle 1 \rangle\}.$$

The formal topology $C = (C, \leq, \lhd)$ is inductively generated by the following axiom

$$u \lhd \{u \frown \langle 0 \rangle, u \frown \langle 1 \rangle\}$$

in the sense that

Thm If $\lhd'$ is another cover relation so that $(C, \leq, \lhd')$ is a formal topology and

$$u \lhd' \{u \frown \langle 0 \rangle, u \frown \langle 1 \rangle\} \qquad (u \in C)$$

then

$$a \lhd U \Longrightarrow a \lhd' U.$$

This gives an induction principle for the cover relation which is an important proof method.

For a given preorder $(X, \leq)$ an *cover axiom* is a pair (a, G) where $a \in X$ and G is a subset of X. We suggestively write this pair as

$$a \vdash G.$$

A formal topology $X = (X, \leq, \lhd)$ is *(inductively)* generated by a family $\{a_i \vdash G_i\}_{i \in I}$ of cover axioms, if $\lhd$ is the smallest cover relation on $(X, \leq)$ which satisfies all the axioms, i.e. $a_i \lhd G_i$ for all $i \in I$.

We may define the formal Baire space as $B = (\mathbb{N}^{<\omega}, \leq, \lhd\,)$ where is $\lhd$ is inductively generated by

$$u \vdash \{u \frown \langle n \rangle : n \in \mathbb{N}\} \qquad\qquad (u \in \mathbb{N}^{<\omega})$$

Proposition: B is a regular formal topology.
Proof: We have $u \lll u$, since $\mathbb{N}^{<\omega} \lhd u \cup u^{\perp}$.

NB: Such covers can be shown to exist using strong induction axioms. In fact for any set of cover axiom, there is a cover relation generated by them.

Set-presented vs inductively generated formal topologies

Def. A formal topology $X = (X, \leq, \lhd)$ is *set-presented* if there is a family of subsets $C(a,i) \subseteq X$ ($a \in X$, $i \in I(a)$) so that

$$a \lhd U \iff (\exists i \in I(a)) C(a,i) \subseteq U.$$

Example

1. Impredicatively, every formal topology X is set-presented by $I(a) = \{U \subseteq X : a \lhd U\}$ and $C(a,U) = U$.
2. The Cantor space is set-presented by

$$\begin{aligned} I(a) &= \{(F,n) \in \mathcal{P}_{\text{f.e.}}(X) \times \mathbb{N} : K(a,n,F)\} \\ C(a,(F,n)) &= F \end{aligned}$$

Theorem. A formal topology is set-presented iff it is inductively generated by some set of cover-axioms. □

Induction on covers

As a simple illustration of the method of induction on covers we give alternative characterisation of points in an inductively generated formal topology $A = (A, \leq, \lhd)$. Suppose that A is generated by the axioms

$$\mathcal{R} = \{a_i \vdash V_i\}_{i \in I}.$$

We call $\alpha \subseteq A$ an $\mathcal{R}$-splitting filter, if it satisfies conditions (i)-(iii) for a point but instead of (iv), only the more restrictive condition that for any $i \in I$

$$a_i \in \alpha \Longrightarrow (\exists b \in V_i)\, b \in \alpha. \tag{1}$$

(This is often easier to check.)

Induction on covers (cont.)

Proposition. Each $\mathcal{R}$-splitting filter is a point.

Proof. We need to prove that for each $\mathcal{R}$-splitting filter α, the condition (iv) holds. The crucial observation is that (iv) can be expressed as:

$$a \lhd U \Longrightarrow [a \in \alpha \Rightarrow (\exists b \in U) b \in \alpha].$$

Since $\lhd$ is the smallest cover relation satisfying $\mathcal{R}$ it is now sufficient to prove that

$$a \lhd' U \Longleftrightarrow_{\mathrm{def}} [a \in \alpha \Rightarrow (\exists b \in U) b \in \alpha].$$

defines a cover relation satisfying $\mathcal{R}$.
By (1) it clearly satisfies $\mathcal{R}$.

Induction on covers (cont.)

It remains to check that $\lhd'$ satisfies (Ext), (Refl), (Trans) and (Loc).
(Ref) is immediate. (Ext) follows by (ii). (Loc) follows by (iii).
(Trans): Suppose $a \lhd' U$ and $U \lhd' W$. We want to show $a \lhd' W$: suppose $a \in \alpha$. Hence there is $b \in U$ with $b \in \alpha$. Thus $b \lhd' W$, and hence $(\exists c \in W) c \in \alpha$, which was to be shown. □

For an inductively generated formal topology A, the collection of points $\mathrm{Pt}(A)$ is in generally only a class.

A point-wise cover which is not a formal cover.

In realizability models the so-called Church Thesis (better named Church-Turing Thesis) is valid. It says that every function on natural numbers is computable by a Turing machine:

$$\text{(CT)} \qquad (\forall f : \mathbb{N} \longrightarrow \mathbb{N}) \\ (\exists e \in \mathbb{N})(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(T(e,x,y) \wedge U(y) = f(x)).$$

Here T is the Kleene predicate and U is the result extracting function.

Assuming (CT) we shall prove that there is a set of basic opens $V \subseteq C$ so that $\mathcal{C} = \bigcup_{v \in V} \mathrm{B}_v$ but $C \lhd V$ fails.

Let

$$V = V_0 \cup V_1,$$

where

$$V_k = \{\langle a_0, \ldots, a_{n-1}\rangle : (\exists i, j < n)(a_i = k \wedge T(i, i, j) \wedge U(j) = k)\}.$$

Note that V_k is decidable and $(V_k)_\leq \subseteq V_k$.
To prove $\mathcal{C} \subseteq \bigcup_{v \in V} \mathrm{B}_v$, take an arbitrary $f \in \mathcal{C}$. We show $f \in \mathrm{B}_v$ for some $v \in V$. Let e be the Kleene index for f, i.e.

$$f(n) = k \Leftrightarrow (\exists y) T(e, n, y) \wedge U(y) = k. \tag{2}$$

We evaluate $f(e)$. Suppose $T(e, e, u)$. We have $f \in B_v$ for $v = \langle f(0), \ldots f(e+u)\rangle$. For $f(u) = k \in \{0, 1\}$ we have $U(u) = k$ by (2). Then by definition $v \in V_k$, so $v \in V$ as required.

Suppose $C \lhd V$. By definition there is a finite $F \subseteq V$ and m so that $w \in F_\leq$ for every w with $|w| \geq m$. (In particular $C \lhd F$.) We show this is impossible.

Define a finite sequence $w = \langle d_0, \ldots, d_{m-1} \rangle \in C$ by letting

$$d_i = 1 \Longleftrightarrow_{\text{def}} i < m \wedge (\exists j < m)(T(i,i,j) \wedge U(j) = 0). \qquad (3)$$

Then $w \leq v$ for some $v \in F$. Since $F \subseteq V = V_{\leq}$ we have $w \in V = V_0 \cup V_1$. Suppose $w \in V_0$. By (2) there are $i, j < m$ with $d_i = 0$, $T(i,i,j)$ and $U(j) = 0$. According to (3) this implies $d_i = 1$ which is a contradiction. Thus $w \in V_1$. Hence there are $i, j < m$ with $d_i = 1$, $T(i,i,j)$ and $U(j) = 1$. But by (3) it follows that $U(j) = 0$. A contradiction again. Hence $C \lhd V$ is false.

The example is a straightforward adaptation of a Theorem which shows that the Fan Theorem is incompatible with CT. The original result is due to Kleene.

This example shows that the Fundamental Adjunction

$$\Omega \dashv \mathrm{Pt} : \mathbf{Top} \longrightarrow \mathbf{Loc}$$

cannot be used without further ado to transfer results from spaces to locales in a constructive setting.

It is necessary to develop point-free techniques to obtain constructive results.

Point-free proofs are more basic

Theorem (Johnstone 1981): Tychonov's Theorem holds for locales, without assuming AC.

Slogan of B. Banaschewski:

choice-free localic argument
+ suitable choice principles = classical result on spaces

In fact, Tychonov's theorem for locales is constructive also in the predicative sense (Coquand 1992).

This suggestive slogan can be generalised

constructive localic argument
+ Brouwerian principle = intuitionistic result on spaces.

However, since the work of the Bishop school (BISH) on constructive analysis it is known that there is often a basic

BISH constructive argument
+ Brouwerian principle = intuitionistic result on *metric spaces.*

A natural question: how does BISH constructive topology and constructive locale theory relate? For instance on metric spaces.

Formal reals $\mathcal{R}$

The basic neighbourhoods of $\mathcal{R}$ are $\{(a, b) \in \mathbb{Q}^2 : a < b\}$ given the inclusion order (as intervals), denoted by $\leq$. The cover $\lhd$ is generated by

(G1) $(a, b) \vdash \{(a', b') : a < a' < b' < b\}$ for all $a < b$,

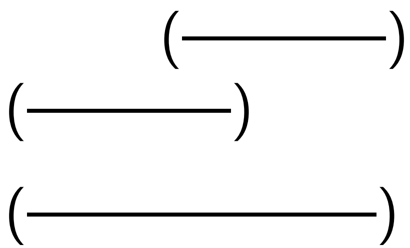
(G2) $(a, b) \vdash \{(a, c), (d, b)\}$ for all $a < d < c < b$.

The set of points $\mathrm{Pt}(\mathcal{R})$ of $\mathcal{R}$ form a structure isomorphic to the Cauchy reals $\mathbb{R}$. For a point α with $(a, b) \in \alpha$ we have by (G2) e.g.

$$(a, (a + 2b)/3) \in \alpha \text{ or } ((2a + b)/3, b) \in \alpha.$$

G1:

G2:



An elementary characterisation of the cover relation on formal reals is:

$$(a,b) \lhd U \iff (\forall a', b' \in \mathbb{Q})(a < a' < b' < b \Rightarrow (\exists \text{ finite } F \subseteq U)\,(a',b')^* \subseteq F^*)$$

Note that for finite F the pointwise inclusion $(a',b')^* \subseteq F^*$ is decidable, since the end points of the intervals are rational numbers. (Exercise)
This characterization is crucial in the proof of the Heine-Borel theorem (see Cederquist, Coquand and Negri).

Closed subspaces

Let X be a formal topology. Each subset $V \subseteq X$ determines a closed sublocale $X \setminus V$ whose covering relation $\lhd'$ is given by

$$a \lhd' U \iff_{\mathrm{def}} a \lhd U \cup V.$$

Note that $V \lhd' \emptyset$ and $\emptyset \lhd' V$, so the new cover relation identifies the open set V with the empty set. What remains is the complement of V.

Theorem If X is compact, and $U \subseteq X$ a subset of neighbourhood, then $X \setminus U$ is compact.

We define $[0,1] = \mathcal{R} \setminus V$ where $V = \{(a,b) : b < 0 \text{ or } 1 < a\}$.
Denote the cover relation of $[0,1]$ by $\lhd'$.

Heine-Borel theorem $[0,1]$ is a compact formal topology.

Proof. Suppose $X \lhd' U$, where X is the basic nbhds of $\mathcal{R}$. Thus in particular $(0-2\varepsilon, 1+2\varepsilon) \lhd' U$ and thus $(0-2\varepsilon, 1+2\varepsilon) \lhd V \cup U$. By the elementary characterisation of $\lhd$ there is a finite $F \subseteq V \cup U$ with $(0-\varepsilon, 1+\varepsilon)^* \subseteq F^*$. As F is finite, we can prove $(0-\varepsilon, 1+\varepsilon) \lhd F$ using G1 and G2. Also we find finite $F_1 \subseteq F \cap U \subseteq U$ so that $(0-\varepsilon, 1+\varepsilon) \lhd V \cup F_1$. Hence $(0-\varepsilon, 1+\varepsilon) \lhd' F_1$. But $X \lhd' (0-\varepsilon, 1+\varepsilon)$ so $X \lhd' F_1$. □

Open subspaces

Let X be a formal topology. Each subset $V \subseteq X$ determines an open sublocale $X_{|V}$ whose covering relation $\lhd'$ is given by

$$a \lhd' U \iff_{\mathrm{def}} a \wedge V \lhd U.$$

Note that $U_1 \lhd' U_2$ iff $U_1 \wedge V \lhd U_2 \wedge V$. Hence only the part inside V counts when comparing two open sets.

Example

For two points $\alpha, \beta \in \mathcal{R}$ the open interval (α, β) is $\mathcal{R}_{|V}$ where

$$V = \{(a, b) : \alpha < a \mathbin{\&} b < \beta\}$$

where $\alpha < a$ means that there is $(c, d) \in \alpha$ with $d < a$. ($b < \beta$ is defined analogously.)

Continuous maps relate the covers

Let $\mathcal{X} = (X, \leq, \lhd)$ and $\mathcal{Y} = (Y, \leq', \lhd')$ be formal topologies. A relation $F \subseteq X \times Y$ is a *continuous mapping* $\mathcal{X} \longrightarrow \mathcal{Y}$ if

- $U \lhd' V \implies F^{-1}U \lhd F^{-1} V$, ("preservation of arbitrary sups")
- $X \lhd F^{-1} Y$, ("preservation of finite infs")
- $a \lhd F^{-1}V,\ a \lhd F^{-1}W \implies a \lhd F^{-1}(V \wedge W)$.
- $a \lhd U$, $x\,F\,b$ for all $x \in U \implies a\,F\,b$,

Each such induces a continuous point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) R(a, b)\} : \mathrm{Pt}(\mathcal{X}) \longrightarrow \mathrm{Pt}(\mathcal{Y})$$

and that satisfies: $a\,F\,b \Rightarrow f[a^*] \subseteq b^*$.

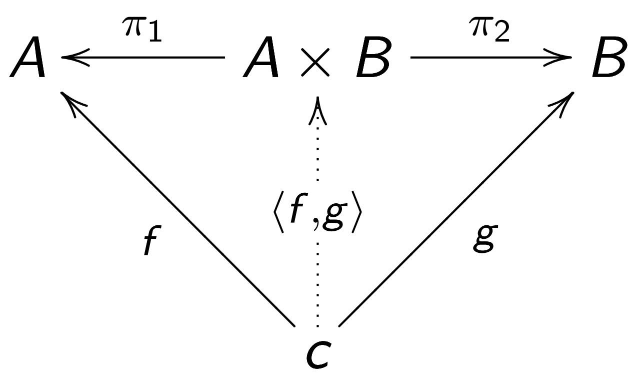
The category of formal topologies.

The inductively generated formal topologies and continuous mappings form a category **FTop**, which is classically equivalent to **Loc**.

They both share many abstract properties with the category of topological spaces which can be expressed in category theoretic language.

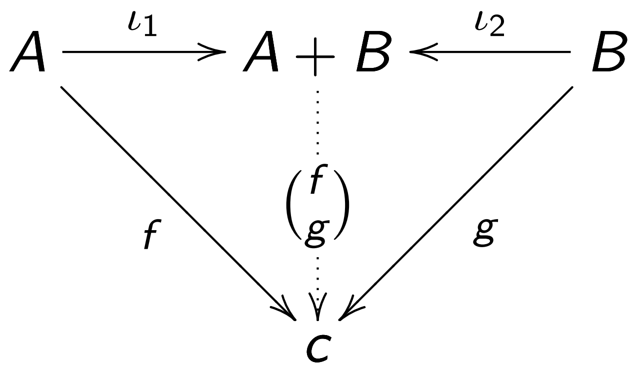
This makes it possible to describe topological constructions without mentioning points.

Recall: A *product* of objects A and B in a category $\mathcal{C}$ is an object $A \times B$ and two arrows $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ in $\mathcal{C}$, so that for any arrows $f : P \longrightarrow A$ and $g : P \longrightarrow B$ there is a unique arrow $\langle f, g\rangle : P \longrightarrow A \times B$ so that $\pi_1\langle f, g\rangle = f$ and $\pi_2\langle f, g\rangle = g$.

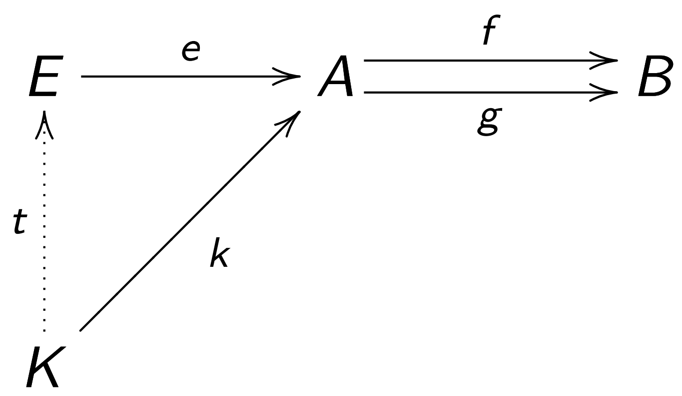


The *dual notion* of product is coproduct or sum.

A *coproduct* of objects A and B in a category $\mathcal{C}$ is an object $A+B$ and two arrows $\iota_1 : A \longrightarrow A+B$ and $\iota_2 : B \longrightarrow A+B$ in $\mathcal{C}$, so that for any arrows $f : A \longrightarrow P$ and $g : B \longrightarrow P$ there is a unique arrow $\binom{f}{g} : A+B \longrightarrow P$ so that $\binom{f}{g}\iota_1 = f$ and $\binom{f}{g}\iota_2 = g$.



A *equalizer* of a pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is a object E and an arrow $e : E \longrightarrow A$ with $fe = ge$ so that for any arrow $k : K \longrightarrow A$ with $fk = gk$ there is a unique arrow $t : K \longrightarrow E$ with $et = k$.



In **Top**: $E = \{x \in A : f(x) = g(x)\}$.

Dual notion: A *coequalizer* of a pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is a object Q and an arrow $q : B \longrightarrow Q$ with $qf = qg$ so that for any arrow $k : B \longrightarrow Q$ with $kf = kg$ there is a unique arrow $t : Q \longrightarrow K$ with $tq = k$.

Categorical topology. As the category of topological spaces has limits and colimits, many spaces of interest can be built up using these universal constructions, starting from the real line and intervals.
The circle

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

is an equaliser of the constant 1 map and $(x, y) \mapsto x^2 + y^2$.
It can also be constructed as a coequaliser of $s, t : \{\star\} \longrightarrow [0, 1]$ where $s(\star) = 0, t(\star) = 1$. (Identifying ends of a compact interval.)

The categorical properties of the category **FTop** of set-presented formal topologies ought to be same as that of the category of locales **Loc**.

Theorem. **Loc** has small limits and small colimits.

However, since we are working under the restraint of predicativity (as when the meta-theory is Martin-Löf type theory) this is far from obvious. (Locales are *complete* lattices with an infinite distributive law.)

FTop has . . .

- Products
- Equalisers (straightforward, ind. gen. covers)
- (and hence) Pullbacks
- Coproducts (Sums)
- Coequalisers
- (and hence) Pushouts
- certain exponentials (function spaces): X^I, when I is locally compact.

Further constructions using limits and colimits

1. The torus may be constructed as the coequaliser of the followings maps $\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$

$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} \qquad (\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} + \mathbf{n}.$$

2. The real projective space $\mathbb{RP}^n$ may be constructed as coequaliser of two maps

$$\mathbb{R}^{n+1} \times \mathbb{R}_{\neq 0} \longrightarrow \mathbb{R}^{n+1}$$

$$(\mathbf{x}, \lambda) \mapsto \mathbf{x} \qquad (\mathbf{x}, \lambda) \mapsto \lambda\mathbf{x}.$$

3. For $A \hookrightarrow X$ and $f : A \longrightarrow Y$, the pushout gives the attaching map construction:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & Y \\ \big\downarrow & & \vdots \\ X & \cdots\cdots\to & Y \cup_f X \end{array}$$

4. The special case of 3, where $Y = 1$ is the one point space, gives the space X/A where A in X is collapsed to a point.

Products of formal topologies

Let $A = (A, \leq_A, \lhd_A)$ and $B = (B, \leq_B, \lhd_B)$ be inductively generated formal topologies. The product topology is $A \times B = (A \times B, \leq', \lhd')$ where

$$(a, b) \leq' (c, d) \Longleftrightarrow_{\mathrm{def}} a \leq_A c \;\&\; b \leq_B d$$

and $\lhd'$ is the smallest cover relation on $(A \times B, \leq_{A \times B})$ so that

- $a \lhd_A U \Longrightarrow (a, b) \lhd' U \times \{b\}$,
- $b \lhd_B V \Longrightarrow (a, b) \lhd' \{a\} \times V$.

The projection $\pi_1 : A \times B \longrightarrow A$ is defined by

$$(a, b)\, \pi_1\, c \Longleftrightarrow_{\mathrm{def}} (a, b) \lhd' A \times \{c\}$$

(Second projection is similar.)

Example

The formal real plane $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}$ has by this construction formal rectangles $((a, b), (c, d))$ with rational vertices for basic neighbourhoods (ordered by inclusion). The cover relation may be characterized in a non-inductive way as

$$
\begin{aligned}
&((a, b), (c, d)) \lhd' U \iff \\
&\quad (\forall u, v, x, y)[a < u < v < b, c < x < y < d \Rightarrow \\
&\qquad (\exists \, \text{finite } F \subseteq U)(u, v) \times (x, y) \subseteq F^*]
\end{aligned}
$$

Subspaces defined by inequations

Let $V \subseteq \mathcal{R}^2$ be the set of open neighbourhoods aboove the graph $y = x$ i.e.

$$V = \{((a,b),(c,d)) \in \mathcal{R}^2 : b < c\}$$

Then $L = (\mathcal{R}^2)_{|V} \hookrightarrow \mathcal{R}^2$ has for points pairs (α, β) with $\alpha < \beta$.
Let $f, g : X \longrightarrow \mathcal{R}$ be continuous maps. Then form the pullback:

$$\begin{array}{ccc} S & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \langle f, g \rangle} \\ L & \hookrightarrow & \mathcal{R}^2 \end{array}$$

Then the subspace $S \hookrightarrow X$ has for points those $\xi \in \mathrm{Pt}(X)$ where $f(\xi) < g(\xi)$.

Coequalizers in formal topology

As the category **Loc** is opposite of the category **Frm** the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in **Loc** can be constructed as the equalizer of the pair $B \overset{f^*}{\underset{g^*}{\rightrightarrows}} A$ in **Frm**.

As **Frm** is "algebraic" the equalizer can be constructed as

$$E = \{b \in B : f^*(b) = g^*(b)\} \hookrightarrow B \overset{f^*}{\underset{g^*}{\rightrightarrows}} A$$

Coequalizers in formal topology (cont.)

A direct translation into **FTop** yields the following suggestion for a construction:

$$\{U \in \mathcal{P}(B) : \widetilde{F^{-1}U} = \widetilde{G^{-1}U}\}$$

for a pair of continuous mappings $F, G : A \longrightarrow B$ between formal topologies. Here $\widetilde{W} = \{a \in A : a \lhd W\}$.

Problem: the new basic neighbourhoods U do not form a set.

It turns out that can one find, depending on F and G, a set of subsets $\mathcal{R}(B)$ with the property that

$$\widetilde{F^{-1}U} = \widetilde{G^{-1}U}, b \in U \Longrightarrow$$
$$(\exists V \in \mathcal{R}(B))(b \in V \subseteq U \; \& \; \widetilde{F^{-1}V} = \widetilde{G^{-1}V})$$

Now the formal topology, whose basic neighbourhoods are

$$Q = \{V \in \mathcal{R}(B) : \widetilde{F^{-1}V} = \widetilde{G^{-1}V}\},$$

and where

$U \lhd_Q W$ iff $U \lhd_B \cup W$

for $W \subseteq \mathcal{R}(B)$, defines a coequaliser. Moreover the coequalising morphism $P : B \longrightarrow Q$ is given by:

$a \, P \, U$ iff $a \lhd_B U$.

References

- P.T. Johnstone. *Stone Spaces.* Cambridge University Press 1982.
- P.T. Johnstone. The point of pointless topology. *Bull. Amer. Math. Soc.* 8(1983), pp. 41 – 53.
- P. Martin-Löf. *Notes on Constructive Mathematics.* Almkvist & Wiksell. Stockholm 1970.
- P. Aczel. Aspects of general topology in constructive set theory. *Ann. Pure Appl. Logic* 137(2006), 3 – 29.
- G. Sambin. Intuitionistic formal spaces — a first communication. In: D. Skordev (ed.) *Mathematical logic and its Applications,* Plenum Press 1987, pp. 187 – 204.

- G. Sambin. Some points in formal topology. *Theoretical Computer Science* 305 (2003), 347 – 408.
- F. Borceux. *Handbook of Categorical Algebra 3: Categories of Sheaves.* Cambridge University Press 1994.
- D.S. Bridges and F. Richman. *Varieties of Constructive Mathematics.* Cambridge University Press 1987.
- T. Coquand. An intuitionistic proof of Tychonoff's theorem. *J. Symbolic Logic* 57(1992), pp. 28 – 32.
- E. Palmgren. A constructive and functorial embedding of locally compact metric spaces into locales. *Topology and its Applications,* 154 (2007), 1854 – 1880.
- E. Palmgren. Resolution of the uniform lower bound problem in constructive analysis. *Mathematical Logic Quarterly,* 54 (2008), 65 – 69.

- G. Curi. *Geometry of Observations: some contributions to (constructive) point-free topology.* PhD Thesis, Siena 2004.
- G. Curi. On the collection of points of a formal space. *Ann. Pure Appl. Logic* 137(2006), 126 – 146.
- G. Curi. The points of (locally) compact regular formal topologies. In: P. Schuster, U. Berger and H. Osswald (eds.) *Reuniting the Antipodes*. Kluwer Academic Publishing, pp. 39 – 54.
- M.E. Maietti. Predicative exponentiation of locally compact formal topologies over inductively generated ones. In: L. Crosilla and P. Schuster (eds.), *From Sets and Types to Topology and Analysis: Towards Practicable Foundations of Constructive Mathematics,* Oxford Logic Guides, Oxford University Press 2005.

- S. Negri and D. Soravia. The continuum as a formal space. *Arch. Math. Logic* 38 (1999), no. 7, 423 – 447.
- S. Negri. Continuous domains as formal spaces. *Mathematical Structures in Computer Science.* 12(2002), 19 – 52.
- G. Sambin, S. Valentini and P. Virgili (1996). Constructive domain theory as a branch of intuitionistic pointfree topology. *Theoretical Computer Science,* 159 (1996), 319 – 341.
- I. Sigstam. Formal spaces and their effective presentations. *Arch. Math. Logic* 34(1995), 211 – 246.
- I. Sigstam and V. Stoltenberg-Hansen. Representability of locally compact spaces by domains and formal spaces. *Theoretical Computer Science* 179(1997), 319 – 331.
- V. Stoltenberg-Hansen and J.V. Tucker. Effective algebras. In: S. Abramsky *et al.* Handbook of Logic in Computer Science, vol. 4. Oxford University Press.

- S. Vickers. *Topology via Logic.* Cambridge University Press 1989.
- S. Vickers. Localic completion of generalized metric spaces I. *Theory and Applications of Categories* 14(2005), 328 – 356.
- S. Vickers. *Localic completion of generalized metric spaces II.* Preprint 2004.
- S. Vickers. *Compactness in locales and in formal topology. Annals of Pure and Applied Logic* 137(2006), 413 – 438.
- S. Vickers, Sublocales in formal topology. *Journal of Symbolic Logic* 72(2007), 463–482.

- E. Palmgren. Maximal and partial points in formal spaces. *Ann. Pure Applied Logic.* To appear.
- E. Palmgren. On universes in type theory. In: G. Sambin and J. Smith (eds.) Twenty-Five Years of Constructive Type Theory. Cambridge Univ. Press 1998.