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Introduction to Combinatorial Homotopy Theory

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Ictp Map Summer School

1 Introduction.

Homotopy theory is a subdomain of topology where, instead of considering the category of topological spaces and continuous *maps*, you prefer to consider as morphisms only the continuous maps *up to homotopy*, a notion precisely defined in these notes in Section 4. Roughly speaking, you decide not to distinguish two maps which can be continuously deformed into each other; such a weakening of the notion of map is quickly identified as necessary when you intend to apply to Topology the methods of *Algebraic Topology.* Otherwise the main classification problems of topology are, except in low dimensions, out of scope.

If you want to "algebraize" the topological world, you will meet another difficulty. The traditional topological spaces, defined for example through collections of open subsets, cannot be directly processed by a computer; a computer can handle only *discrete* objects and in a sense topology is the opposite subject. A *combinatorial* intermediary notion between Topology and Algebra is required. Poincaré started Algebraic Topology about a century ago by using the polyhedra as intermediary objects, but since the fifties, the *simplicial* notions have been recognized as more appropriate. In this framework of combinatorial topology, the sensible topological spaces can be combinatorially defined, and also installed and processed on a computer. This is valid even for very complicated or abstract spaces such as classifying spaces, functional spaces; the various important functors of algebraic topology can also be implemented as functional objects.

In a sense there is a conflict between both previous observations. The homotopy relation is concerned by *continuous* deformations of maps, while combinatorial models for topological spaces *and* maps do not seem to allow enough maps to model homotopies. But we will see this apparent obstacle is easily overcome, and the so-called *Combinatorial Homotopy Theory* is now one of the standard ground theories for Algebraic Topology.

In particular, if you claim you are mainly interested by *constructive* results in Algebraic Topology, it is quickly obvious *combinatorial* topology is required. *Constructive Algebraic Topology* is a difficult but fascinating subject, and three main solutions are now available:

1. Rolf Schön's solution [21], quite elegant, unfortunately never (?) considered since his remarkable memoir, in particular from a concrete programming point of view.

2. The solution studied for years by this author and several collaborators, see the lecture notes of the previous Map Summer School at Genova [20]. The key point is that *locally effective* models for combinatorial spaces are sufficient to use standard simple Algebraic Topology and make it constructive.

3. The *operadic* solution where the algebraic world is enriched enough to make it equivalent to the topological world; more precisely the algebraic structure of chain complexes is sufficiently enriched, thanks to appropriate *operads*, to code in this way the topological spaces up to homotopy.

But whatever solution you decide to study, anyway you will have to use ingredients coming from combinatorial homotopy. For the solutions 1 and 2 above, it will even be necessary to implement on your computer the corresponding necessary simplicial objects and operators; for the solution 3, the objects of the resulting category, the E_∞-operadic chain complexes, do not seem to use combinatorial homotopy, but the theoretical justifications requires by some means or other this theory. This Ictp-Map Summer School proposes an introduction to the solution 3 and the present lecture is intended to prepare the audience to the most elementary facts of combinatorial homotopy.

Section 2 describes the most elementary simplicial techniques, around the notion of simplicial *complex*. It is already possible in this simple framework to speak of combinatorial homotopy, for example it is possible to construct simplicial models for functional spaces, in particular for loop spaces. An important progress at the end of the forties was the invention (discovery?), mainly by Samuel Eilenberg, of the notion of simplicial *set*, to which the rest of these notes is devoted. An amusing paradox of this terminology must be signaled: the notion of simplicial *set* is much more complex than the notion of simplical… complex! These simplicial sets were initially called CSS-sets, an acronym for "complete-semi-simplicial"; but it was identified a little later the general notion of *simplicial object* in an arbitrary category makes sense and a CSS-set is nothing but a simplicial object in the category of sets, which explains the modern and natural terminology of *simplicial set*.

This notion of simplicial set is one of the most fascinating elementary notions in mathematics. In a sense it contains the whole richness of topology. Yet an essential drawback must immediately be pointed out: modelling a topological object as a simplicial set leads to coherent but arbitrary choices of orders (resp. orientations) for the vertices (resp. simplices). It happens these choices hide very sophisticated actions of the symmetric groups $\mathfrak{S}_n$; in a sense, elementary Algebraic Topology forgets this action and *operadic* Algebraic Topology on the contrary takes account of this action, in a totally algebraic framework, and in this way, the initial goal of Algebraic Topology, representing homotopy types as algebraic objects, is finally reached.

Once the notion of simplicial set is available, most ingredients of algebraic topology, classifying spaces, loop spaces, functional spaces, homology or cohomology groups, any sort of operators between these groups can be more or less easily described in the framework of simplicial sets. The initial essential step in this direction was the discovery by Daniel Kan [12] of a purely combinatorial definition of homotopy groups. The end of these notes shows a few typical examples of simplicial descriptions, mainly to prepare the readers to the lecture about Operadic Algebraic Topology.

Contents

2 Simplicial complexes.

2.1 Definitions.

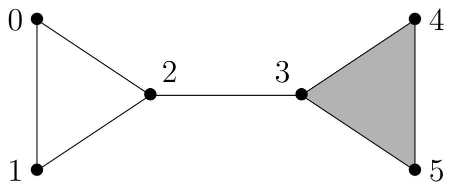
Definition 1 — A *simplicial complex* is a pair (V, S) satisfying the properties:

- V, the set of *vertices*, and S, the set of *simplices*, are... sets, possibly infinite.
- Every simplex $\sigma \in S$ is a non-empty finite set of vertices: $\sigma = \{v_0, \ldots, v_n\}$; such a simplex is called an n-simplex, the integer $n \geq 0$ is the *dimension* of the simplex σ. This simplex *spans* the vertices $v_0, \ldots, v_n$.
- For every vertex $v \in V$, the 0-simplex $\{v\}$ is an element of S.
- For every simplex $\sigma = \{v_0, \ldots, v_n\} \in S$, every m-sub-simplex $\{v_{i_0}, \ldots, v_{i_m}\}$ is also an element of S.

For example, let us consider the simplicial complex (V, S) with:

- $V = \{0 \ldots 5\}$, the integers from 0 to 5.
- $S = \{0, 1, 2, 3, 4, 5, 01, 02, 12, 23, 34, 35, 45, 345\}$ where 35 for example is a shorthand for $\{3, 5\}$.

Such a simplicial complex is an "abstract" version of the geometrical object:



The triangle 012 is *hollow*, because $\{0, 1, 2\}$ is not a simplex; on the contrary, $\{3, 4, 5\}$ is a simplex and the triangle 345 is *filled*. In the simplicial complex game, you have a box with an arbitrary number of available vertices (0-simplices), edges (1-simplices), triangles, (2-simplices), tetrahedons (3-simplices) and more generally of n-simplices. Every vertex is labeled by the corresponding element of V and the simplices of S describe what collections of vertices are spanned by a simplex.

No geometry in this definition; in particular, at this level, a simplex is just an "abstract" *set* of vertices, which, when we will geometrically *realize* in a moment a simplicial complex, will finally produce an ordinary geometrical simplex.

Definition 2 — Let $K = (V, S)$ be a simplicial complex. The *geometrical realization* $|K|$ of K is defined as follows: $|K|$ is the set of the indexed families $x = (x_\sigma)_{\sigma \in S} \in [0,1]^{(S)}$ satisfying the conditions:

- $\{\sigma \ \underline{\mathbf{st}}\ x_\sigma > 0\} \in S$ and in particular is finite;
- $\sum x_\sigma = 1$.

Any topology over $[0,1]^{(S)}$ defines a topology over $|K|$, but *combinatorial topology* most often is not concerned by such a topology: the combinatorial game is enough to model, *up to homotopy*, in this way most "sensible" topological spaces.

2.2 Simple examples.

Let V be an arbitrary set of vertices, possibly infinite. Then the *simplex generated by* V, denoted by Δ^V, is the simplicial complex (V, S) where $S = \mathcal{P}_{*,f}(V)$ is the set of *finite* non-empty subsets of V. If $V = \underline{n} := \{0, 1, \ldots, n\}$, then $\Delta^{\underline{n}}$ is usually simply denoted by $\Delta^n = (\underline{n}, \mathcal{P}_*(\underline{n}))$, it is the standard (abstract) n-simplex, and its realization $|\Delta^n|$ is the common geometrical n-simplex. If V is infinite, then the simplicial complex Δ^V has simplices of arbitrary high dimension, but every simplex of Δ^V has a finite dimension.

The standard model for the n-sphere S^n as a simplicial complex is:

$$S^n = (\underline{n+1}, \mathcal{P}_*(\underline{n+1}) - \{\underline{n+1}\}). \tag{1}$$

It is the standard $n+1$-simplex Δ^{n+1} from which the maximal simplex $\underline{n+1} = \{0, \ldots, n+1\}$ has been removed: think the standard simplex Δ^{n+1} is *solid* and you may so imagine our model for the n-sphere is on the contrary a *hollow* $(n+1)$-simplex, in other words the boundary of an $(n+1)$-simplex. Its realization is homeomorphic to the boundary of an $(n+1)$-disk (or cell, or ball), that is, a topological n-sphere.

Many topological constructions can be simulated in the framework of simplicial complexes. For example, if $K = (V, S)$ and $K' = (V', S')$ are two simplicial complexes *with base point*, that is, two vertices $v_0 \in V$ and $v'_0 \in V'$ are distinguished, then the *wedge* $K \vee K'$ is defined by $K'' = (V'', S'')$ with $V'' = (V \coprod V')/(v_0 \sim v'_0)$ and $S'' = (S \coprod S')/\!\sim$ where the last relation $\sim$ identifies any occurence of v_0 in an element of S with any occurence of v'_0 in an element of S'. Both simplicial complexes are "attached" at their respective base vertices.

A common construction is however surprisingly difficult to be translated in the framework of simplicial complexes, namely the *product* construction. The difficulty is the following: the elementary piece in the world of simplicial complexes is a simplex, a point in dimension 0, an edge in dimension 1, a solid triangle in dimension 2, a tetrahedron in dimension 3, an n-simplex in dimension n. But the product of two edges is a square, which can be presented as the union of two triangles, if you cut this square along a diagonal; but two diagonals in a square and how to choose the right one? A little more difficult, the product of an edge by a (solid) triangle is a triangular prism which can be presented as the union of three tetrahedrons, a process neither easy nor deterministic. We will see later the product of an m-simplex by an n-simplex can be divided in $\binom{m+n}{m}$ simplices of dimension $(m+n)$, by a process not so obvious, made "automatic" if you work in the framework of simplicial *sets*.

2.3 Simplicial maps and homotopy.

Definition 3 — Let $K = (\text{V,S})$ and $K' = (V', S')$ be two simplicial complexes. A *simplicial map* $f : K \to K'$ is a set map $f : V \to V'$ satisfying the property: for every simplex $\sigma \in S$, the image $f(\sigma)$ is a simplex $f(\sigma) \in S'$.

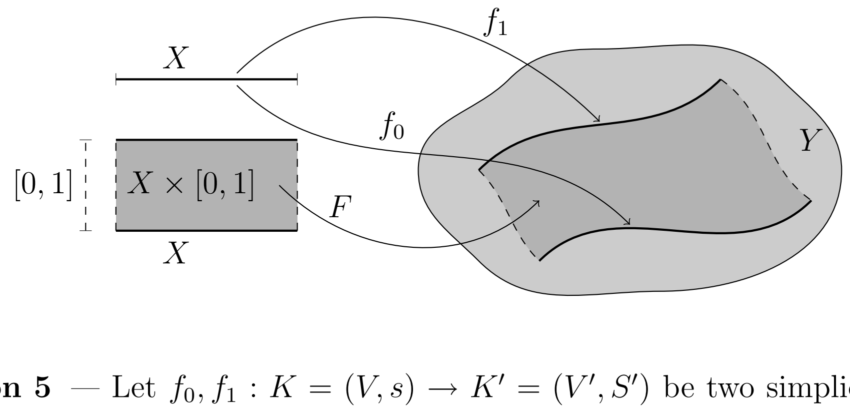
It is not required $f : V \to V'$ is injective, and the image of an n-simplex $\sigma \in S$ can be a simplex $f(\sigma) \in S'$ of dimension $< n$.

Is it possible to define homotopies between simplicial maps? First, let us consider the traditional notion of homotopy between continuous maps.

Definition 4 — Two continuous maps $f_0, f_1 : X \to Y$ between the topological spaces X and Y are *homotopic* if there exists a (continuous) map $F : [0,1] \times X \to Y$ satisfying:

$$\begin{array}{rcl} F(0,x) & = & f_0(x) \\ F(1,x) & = & f_1(x) \end{array} \tag{2}$$

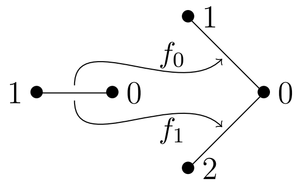
for every $x \in X$.



Definition 5 — Let $f_0, f_1 : K = (V, s) \to K' = (V', S')$ be two simplicial maps between two simplicial complexes. The maps f_0 and f_1 are *elementarily homotopic* if the following property is satisfied: for every simplex $\sigma \in V$, the union $f_0(\sigma) \cup f_1(\sigma)$ is a simplex of S'.

If the required property is satisfied, you can then, at the level of the geometrical realizations, trivially interpolate the maps f_0 and f_1 by a continuous family of f_t's, for t running the interval $[0,1]$. Note f_t cannot be simplicially implemented except for $t = 0$ or 1.

Definition 5 is natural but not at all satisfactory. Let us consider the situation with K the interval $K = \Delta^1 = (\underline{1}, \mathcal{P}_*(\underline{1}))$ and $K' = (\underline{2}, S')$ with S' made of the three vertices $\{0\}$, $\{1\}$ and $\{2\}$, and only two edges $\{0,1\}$ and $\{0,2\}$. Let us consider also the maps $f_0, f_1 : K \to K'$ defined by $f_0(0) = f_1(0) = 0$, $f_0(1) = 1$ and $f_1(1) = 2$. Then these maps are not elementarily homotopic though, in the topological framework, they are homotopic.

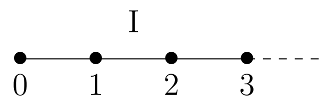


This difficulty can be overcome as follows: you decide two simplicial maps $f, g : K \to K'$ are homotopic if you can construct a *chain* $f = f_0, f_1, \ldots, f_{k-1}, f_k = g$ where two successive elements are elementarily homotopic. We let you construct the simple chain of length 2 describing how the maps of the previous example are homotopic. But for infinite simplicial complexes, such a solution is not satisfactory. The technique of *Kan* simplicial sets allows to overcome this important obstacle, at the cost of complex technicalities, complex but unavoidable.

It is possible also to define *functional spaces* in a combinatorial style. Because the framework of simplicial complexes will be soon given up, we show only a typical example: how to define the loop space of a pointed simplicial complex $(K, *)$, the *base point* $*$ being a *distinguished* vertex of K? Usually a loop $\gamma : [0,1] \to (X, *)$ in a pointed topological space is a continuous map $\gamma : [0,1] \to X$ satisfying $\gamma(0) = \gamma(1) = *$. How to copy this notion for simplicial complexes?

The interval $[0,1]$ is (the realization of) a simplicial complex and the notion of simplicial map $\gamma : [0,1] \to K$ makes sense; but combined with the condition $\gamma(0) = \gamma(1) = *$, only one such loop, the trivial constant loop at the base point, not very satisfactory!

To overcome this obstacle, instead of the simple interval $[0,1]$, let us consider the (infinite) simplicial complex $I = (\mathbb{N}, S)$ with $S = \{\{n\}\}_{n\in\mathbb{N}} \cup \{\{n, n+1\}\}_{n\in\mathbb{N}}$.



First it is natural to decide a loop $\gamma : I \to K$ is a simplicial map satisfying:

- $\gamma(0) = *$;
- For every $n \geq$ some n_0, $\gamma(n) = *$.

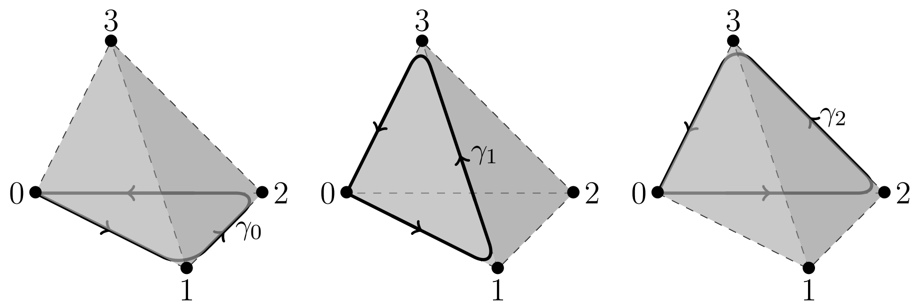
Our loop starts from the base point, runs various edges of K, and after the time n_0, remains fixed at the base point.

Then the loop space ΩK can be naturally defined *as a simplicial complex* as follows: $\Omega K = (\Lambda, S_\Lambda)$ with:

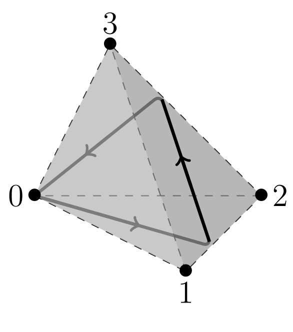
- Λ is the set of loops as just defined;
- A finite set of loops $\{\lambda_0, \ldots, \lambda_n\}$ is an element of S_Λ, that is, a simplex of ΩK, if and only if, for every integer $t > 0$, the set $\{\lambda_0(t-1), \ldots, \lambda_n(t-1)\} \cup \{\lambda_0(t), \ldots, \lambda_n(t)\}$ is a simplex of K.

The last condition claims that it is possible to *interpolate* in a barycentric style the loops $\lambda_0, \ldots, \lambda_n$ for every point of the "geometrical" simplex intuitively spanned by these loops; if the condition is satisfied, we therefore decide to install an "abstract" simplex between these vertices. It is an interesting exercise of topology to prove the realization $|\Omega K|$ actually has the same homotopy type as the (topological) loop space $\Omega(|K|)$, but this will not be necessary in these notes.

For example if $K = S^2$ modelled as the boundary of the standard 3-simplex: $K = (0..3, \{0, 1, 2, 3, 01, 02, 03, 12, 13, 23, 012, 013, 023, 123\})$, let us consider the loops $\gamma_0 = 0 \to 1 \to 2 \to 0$, $\gamma_1 = 0 \to 1 \to 3 \to 0$ and $\gamma_2 = 0 \to 2 \to 3 \to 0$, with notations made obvious by the figure:



Then $\{\gamma_0, \gamma_1\}$, $\{\gamma_0, \gamma_2\}$ and $\{\gamma_1, \gamma_2\}$ are edges of ΩK, and $\{\gamma_0, \gamma_1, \gamma_2\}$ is a triangle of ΩK between these edges; drawing the loop which is the "center" of this triangle is useful.



Note this "loop" is not an actual loop of our simplicial complex: such a loop is allowed to run only the edges of our sphere, while our "loop" goes *inside* some triangles. This claimed "loop" is only a geometric interpretation of the center of the triangle of ΩK spanning the actual loops γ_0, γ_1 and γ_2.

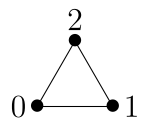
On the contrary, if γ_1^{-1} is the same loop as γ_1 but run in the reverse direction (meaning?), then $\{\gamma_0, \gamma_1^{-1}, \gamma_2\}$ is not a triangle of ΩK, why?

Let us decide the base point $* \in \Omega K$ is the trivial loop constant in 0. Then $* \to \gamma_0 \to \gamma_1 \to *$ is a *loop* of ΩK, in other words a vertex of $\Omega\Omega K =: \Omega^2 K$. Proving the last loop is not homotopic to the trivial loop is another story.

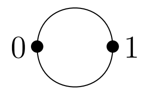
2.4 Simplicial *complexes* vs simplicial *sets*.

A $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ can in particular be a face operator $\partial_i^m : \underline{m-1} \to \underline{m}$. The corresponding X-operator $X_{\partial_i^m} : X_m \to X_{m-1}$ is also called the i-th face operator in dimension m and is most often simply denoted by ∂_i^m or ∂_i when the underlying simplicial set X is implicit. The same for a degeneracy operator $X_{\eta_i^m} : X_m \to X_{m+1}$, most often denoted by η_i^m or η_i. Because of Corollary 11, it is enough to define the face and degeneracy operators $X_{\partial_i^m}$ and $X_{\eta_i^m}$ satisfying the required coherence properties, to define the whole collection of morphisms $\{X_\alpha\}_\alpha$.

Let us consider the simplest simplicial *complex* X, the realization of which is (homeomorphic to) a circle $S^1 = \{(x, y) \in \mathbb{R}^2 \ \underline{\mathbf{st}}\ x^2 + y^2 = 1\}$. Three vertices and three edges are necessary: $X = (V, S)$ with $V = \underline{2} = \{0, 1, 2\}$ and $S = \{0, 1, 2, 01, 02, 12\}$ where as usual 01 is a shorthand for $\{0, 1\}$.

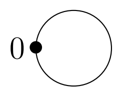


It is not possible to use only two vertices following the figure:



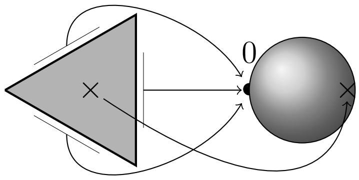
for the only possibility to *produce* an edge consists in choosing a set of vertices; so that it is possible to install only *one* edge between two given vertices and the above figure cannot correspond to a simplicial *complex*. We will see that we do not meet any problem when associating a simplicial *set* to the same figure, this will be explained soon.

We could even consider the following figure:



and observe that is is not possible in the framework of simplicial *complexes* to install a "loop" edge from a vertex to *itself*. This is also possible for simplicial *sets*.

We will see it is also possible to give a simplicial set with only two (non-degenerate) simplices, a vertex 0 and a “triangle” 012, the three edges of which being collapsed over the unix vertex.



The realization of this simplicial set will be a triangle where the whole boundary is identified to a point, that is, a 2-sphere.

More generally any n-sphere can be realized as a simplicial *set* with only one vertex and one n-simplex; more precisely only these *non-degenerate* simplices, for we will soon learn that any non-degenerate simplex generates an infinite collection of… degenerate simplices, non-visible on the figures, that is, “hidden” in the geometric realization. For example the minimal simplicial complex corresponding to a 4-sphere requires 6 vertices, 15 edges, 20 triangles, 15 tetrahedron and 6 4-simplices, while as a simplicial set, only one vertex and one 4-simplex are enough as non-degenerate simplices.

These elementary examples show in general less (non-degenerate) simplices are necessary to construct an object as a simplicial set than as a simplicial complex. You can object an infinite number of degenerate simplices is also required, but precisely these degenerate simplices will give *much more flexibility* in the construction process. It is true the underlying technology is not obvious, but thanks to this nice technology, the main parts of topology have a good translation into the combinatorial world, allowing a constructivist to easily handle topology with his computer.

3 Simplical sets.

Possible references for this fascinating subjects are:

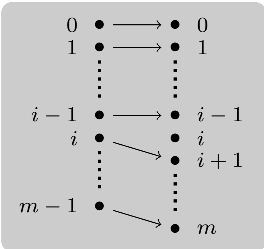
- [13]: Maybe the most useful reference for the serious user; only one drawback: hardly any example, no didactic explanation! But many invaluable formulas and detailed proofs can be found only in this book.
- [15]: See in particular Section VIII.5 of this book for a short introduction to this subject, which is not the main goal of this book, but unavoidable.
- [14]: See Section §4.2.
- More modern, but also harder, references are [10], a book entirely devoted to this subject, and also [9, I.2].

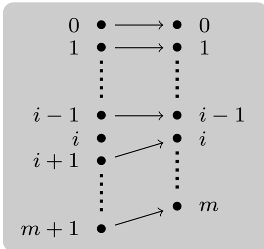
3.1 The category $\underline{\Delta}$.

Some strongly structured sets of indices are necessary to define the notion of *simplicial object*; they are conveniently organized as the category $\underline{\Delta}$. An object of $\underline{\Delta}$ is a set $\underline{m}$, namely the set of integers $\underline{m} := \{0, 1, \ldots, m-1, m\}$; this set is canonically *ordered* with the usual order between integers.

A $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ is an *increasing* map. Equal values are permitted; for example a $\underline{\Delta}$-morphism $\alpha : \underline{2} \to \underline{3}$ could be defined by $\alpha(0) = \alpha(1) = 1$ and $\alpha(2) = 3$. The set of $\underline{\Delta}$-morphisms from $\underline{\text{m}}$ to $\underline{\text{n}}$ is denoted by $\underline{\Delta}(\underline{m}, \underline{n})$; the subset of injective (resp. surjective) morphisms is denoted by $\Delta^{\text{inj}}(\underline{m}, \underline{n})$ (resp. $\Delta^{\text{srj}}(\underline{m}, \underline{n})$).

Some *elementary* morphisms are important, namely the simplest non-surjective and non-injective morphisms. For geometric reasons explained later, the first ones are the *face morphisms*, the second ones are the *degeneracy morphisms*.

$\partial_i^m =$ 

$\eta_i^m =$ 

Definition 6 — The *face morphism* $\partial_i^m : \underline{m-1} \to \underline{m}$ is defined for $m \geq 1$ and $0 \leq i \leq m$ by:

$$\begin{array}{ll} \partial_i^m(j) = & j \quad \text{if } j < i, \\ \partial_i^m(j) = & j+1 \quad \text{if } j \geq i. \end{array} \tag{3}$$

The face morphism ∂_i^m is the unique injective morphism from $\underline{\text{m-1}}$ to $\underline{\text{m}}$ such that the integer i is not in the image. The face morphisms generate the injective morphisms, in fact in a unique way if a growth condition is required.

Proposition 7 — *Any injective $\underline{\Delta}$-morphism $\alpha \in \Delta^{\text{inj}}(\underline{m}, \underline{n})$ has a unique expression:*

$$\alpha = \partial_{i_n}^n \circ \ldots \circ \partial_{i_{m+1}}^{m+1} \tag{4}$$

satisfying the relation $i_n > i_{n-1} > \ldots > i_{m+1}$.

♣ The index set $\{i_{m+1}, \ldots, i_n\}$ is exactly the difference set $\underline{n} - \alpha(\underline{m})$, that is, the set of the integers where surjectivity fails. ♣

Frequently the upper index m of ∂_i^m is omitted because clearly deduced from the context. For example the unique injective morphism $\alpha : \underline{2} \to \underline{5}$ the image of which is $\{0, 2, 4\}$ can be written $\alpha = \partial_5 \partial_3 \partial_1$.

If two face morphisms are composed in the wrong order, they can be exchanged: $\partial_i \circ \partial_j = \partial_{j+1} \circ \partial_i$ if $j \geq i$. Iterating this process allows you to quickly compute for example $\partial_0 \partial_2 \partial_4 \partial_6 = \partial_9 \partial_6 \partial_3 \partial_0$.

Definition 8 — The *degeneracy morphism* $\eta_i^m : \underline{m+1} \to \underline{m}$ is defined for $m \geq 0$ and $0 \leq i \leq m$ by:

$$\begin{aligned} \eta_i^m(j) = & \; j & \text{if } j \leq i, \\ \eta_i^m(j) = & \; j-1 & \text{if } j > i. \end{aligned} \tag{5}$$

The degeneracy morphism η_i^m is the unique surjective morphism from $\underline{\text{m+1}}$ to $\underline{\text{m}}$ such that the integer i has two pre-images. The degeneracy morphisms generate the surjective morphisms, in fact in a unique way if a growth condition is required.

Proposition 9 — *Any surjective $\underline{\Delta}$-morphism $\alpha \in \Delta^{\text{srj}}(\underline{m}, \underline{n})$ has a unique expression:*

$$\alpha = \eta_{i_n}^n \circ \ldots \circ \eta_{i_{m-1}}^{m-1} \tag{6}$$

satisfying the relation $i_n < i_{n+1} < \ldots < i_{m-1}$.

♣ The index set $\{i_n, \ldots, i_{m-1}\}$ is exactly the set of integers j such that $\alpha(j) = \alpha(j+1)$, that is, the integers where injectivity fails. ♣

Frequently the upper index m of η_i^m is omitted because clearly deduced from the context. For example the unique surjective morphism $\alpha : \underline{5} \to \underline{2}$ such that $\alpha(0) = \alpha(1)$ and $\alpha(2) = \alpha(3) = \alpha(4)$ can be expressed $\alpha = \eta_0\eta_2\eta_3$.

If two degeneracy morphisms are composed in the wrong order, they can be exchanged: $\eta_i \circ \eta_j = \eta_j \circ \eta_{i+1}$ if $i \geq j$. Iterating this process allows you to quickly compute for example $\eta_3\eta_3\eta_2\eta_2 = \eta_2\eta_3\eta_5\eta_6$.

Proposition 10 — *Any $\underline{\Delta}$-morphism α can be $\underline{\Delta}$-decomposed in a unique way:*

$$\alpha = \beta \circ \gamma \tag{7}$$

with β injective and γ surjective.

♣ The intermediate $\underline{\Delta}$-object $\underline{\text{k}}$ necessarily satisfies $k + 1 = \text{Card}(\mathbf{im}(\alpha))$. The growth condition then gives a unique choice for β and γ. ♣

Corollary 11 — *Any $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ has a unique expression:*

$$\alpha = \partial_{i_n} \circ \ldots \circ \partial_{i_{k+1}} \circ \eta_{j_k} \circ \ldots \circ \eta_{j_{m-1}} \tag{8}$$

satisfying the conditions $i_n > \ldots > i_{k+1}$ and $j_k < \ldots < j_{m-1}$. ♣

Finally if face and degeneracy morphisms are composed in the wrong order, they can be exchanged:

$$\begin{aligned} \eta_i \circ \partial_j &= \text{id} & \text{if } j = i \text{ or } j = i+1; \\ &= \partial_{j-1} \circ \eta_i & \text{if } j \geq i+2; \\ &= \partial_j \circ \eta_{i-1} & \text{if } j < i. \end{aligned} \tag{9}$$

All these commuting relations can be used to convert an arbitrary composition of faces and degeneracies into the canonical expression:

$$\alpha = \eta_9\partial_6\eta_3\partial_7\eta_9\partial_8\eta_6\partial_2\eta_4\partial_9 = \partial_7\partial_6\partial_2\eta_2\eta_4\eta_6. \tag{10}$$

This relation means the image of α does not contain the integers 2, 6 and 7, and the relations $\alpha(2) = \alpha(3)$, $\alpha(4) = \alpha(5)$ and $\alpha(6) = \alpha(7)$ are satisfied.

The previous propositions show any functor F from $\underline{\Delta}$ to another category is entirely known when the image objects $F(\underline{m})$ and the image morphisms $F(\partial_i^m)$ and $F(\eta_i^m)$ are given.

Corollary 12 — *A* contravariant *functor* $X : \underline{\Delta} \to \underline{\mathrm{CAT}}$ *is nothing but a collection* $\{X_m\}_{m\in\mathbb{N}}$ *of objects of the target category* $\underline{\mathrm{CAT}}$*, and collections of* $\underline{\mathrm{CAT}}$*-morphisms* $\{X(\partial_i^m) : X_m \to X_{m-1}\}_{m\geq 1\,,0\leq i\leq m}$ *and* $\{X(\eta_i^m) : X_m \to X_{m+1}\}_{m\geq 0\,,0\leq i\leq m}$ *satisfying the commuting relations:*

$$\begin{array}{rcll} X(\partial_i)\circ X(\partial_j) & = & X(\partial_j)\circ X(\partial_{i+1}) & \text{if } i \geq j, \\ X(\eta_i)\circ X(\eta_j) & = & X(\eta_{j+1})\circ X(\eta_i) & \text{if } j \geq i, \\ X(\partial_i)\circ X(\eta_j) & = & \mathrm{id} & \text{if } i = j, j+1, \\ X(\partial_i)\circ X(\eta_j) & = & X(\eta_{j-1})\circ X(\partial_i) & \text{if } j > i, \\ X(\partial_i)\circ X(\eta_j) & = & X(\eta_j)\circ X(\partial_{i-1}) & \text{if } i > j+1. \end{array} \tag{11}$$

In the five last relations, the upper indices have been omitted. Such a contravariant functor is a *simplicial object* in the category $\underline{\mathrm{CAT}}$. If α is an arbitrary $\underline{\Delta}$-morphism, it is then sufficient to express α as a composition of face and degeneracy morphisms; the image $X(\alpha)$ is necessarily the composition of the images of the corresponding $X(\partial_i)$'s and $X(\eta_i)$'s; the above relations ensure the definition is coherent.

3.2 Simplicial sets: first definitions.

Definition 13 — Let $\underline{\mathrm{Set}}$ be the category of sets. A *simplicial set* X is a simplicial object in the $\underline{\mathrm{Set}}$ category; that is, according to the previous section, a contravariant functor $X : \underline{\Delta} \to \underline{\mathrm{Set}}$.

This definition is short, but, because of the rich structure of the category $\underline{\Delta}$, it is quite complex! You see defining a simplicial set X requires for every nonnegative integer n some object X_n of the $\underline{\mathrm{Set}}$ category, in other words an ordinary set, and for every $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$ some $\underline{\mathrm{Set}}$-morphism, that is an ordinary map $X_\alpha : X_n \to X_m$; furthermore, the set $\{X_\alpha\}_{\alpha\in\underline{\Delta}-\mathrm{morphisms}}$ must satisfy the coherence relations $X_\alpha X_\beta = X_{\beta\alpha}$ when the composition $\beta\alpha$ makes sense.

The geometric interpretation of this definition is not obvious, but once understood, this notion is terribly powerful. A power which deserves a little significant work to reach its marvelous possibilities. Before seriously studying this notion, let us give a few comments about the comparison between simplicial *complexes* and simplicial *sets*.

A *simplicial set* X is a simplicial object in the category of sets, and therefore is given by a collection of sets $\{X(\underline{m})\}_{m\in\mathbb{N}}$ and collections of maps $\{X_\alpha\}$, the index α running the $\underline{\Delta}$-morphisms; the usual coherence properties must be satisfied. As explained at the end of Section 4, it is sufficient to define the $X(\partial_i^m)$'s and the $X(\eta_i^m)$'s with the corresponding commuting relations.

The set $X(\underline{m})$ is usually denoted by X_m and is called the set of m-simplices of X; such a simplex has the *dimension* m. To be a little more precise, these simplices are sometimes called *abstract* simplices, to avoid possible confusions with the *geometric* simplices defined a little later. An (abstract) m-simplex is only *one* element of X_m.

If $\alpha \in \underline{\Delta}(\underline{n}, \underline{m})$, the corresponding morphism $X(\alpha) : X_m \to X_n$ is most often simply denoted by $\alpha^* : X_m \to X_n$ or still more simply $\alpha : X_m \to X_n$. In particular the faces and degeneracy operators are maps $\partial_i : X_m \to X_{m-1}$ and $\eta_i : X_m \to X_{m+1}$. If σ is an m-simplex, the (abstract) simplex $\partial_i\sigma$ is its i-th face, and the simplex $\eta_i\sigma$ is its i-th degeneracy; we will see the last one is "particularly" abstract.

3.3 The structure of simplex sets.

Definition 14 — An m-simplex σ of the simplicial set X is *degenerate* if there exist an integer $n < m$, an n-simplex $\tau \in X_n$ and a $\underline{\Delta}$-morphism $\alpha \in \underline{\Delta}(\underline{m}, \underline{n})$ such that $\sigma = \alpha(\tau)$. The set of non-degenerate simplices of dimension m in X is denoted by X_m^{ND}.

Decomposing the morphism $\alpha = \beta \circ \gamma$ with γ surjective, we see that $\sigma = \gamma(\beta(\tau))$, with the dimension of $\beta(\tau)$ less or equal to n; so that in the definition of degenaracy, the connecting $\underline{\Delta}$-morphism α can be required to be surjective. The relation $\sigma = \alpha(\tau)$ with α surjective is shortly expressed by saying the m-simplex σ *comes from* the n-simplex τ.

Eilenberg's lemma explains each degenerate simplex comes from a canonical non-degenerate one, and in a unique way.

Lemma 15 — **(Eilenberg's lemma)** *If X is a simplicial set and σ is an m-simplex of X, there exists a unique triple $T_\sigma = (n, \tau, \alpha)$ satisfying the following conditions:*

1. *The first component n is a natural number $n \leq m$;*
2. *The second component τ is a non-degenerate n-simplex $\tau \in X_n^{ND}$;*
3. *The third component α is a $\underline{\Delta}$-morphism $\tau \in \Delta^{\mathrm{srj}}(\underline{m}, \underline{n})$;*
4. *The relation $\sigma = \alpha(\tau)$ is satisfied.*

Definition 16 — This triple T_σ is called the *Eilenberg triple* of σ.

♣ Let $\mathcal{T}$ be the set of triples $T = (n, \tau, \alpha)$ such that $n \leq m$, $\tau \in X_n$ and $\alpha \in \underline{\Delta}(\underline{m}, \underline{n})$ satisfy $\sigma = \alpha(\tau)$. The set $\mathcal{T}$ certainly contains the triple (m, σ, id)

and therefore is non empty. Let (n_0, τ_0, α_0) be an element of $\mathcal{T}$ where the first component, the integer n_0, is minimal. We claim (n_0, τ_0, α_0) is the Eilenberg triple.

Certainly $n_0 \leq m$. The n_0-simplex τ_0 is non-degenerate; otherwise $\tau_0 = \beta(\tau_1)$ with the dimension n_1 of τ_1 less than n_0, but then $(n_1, \tau_1, \beta\alpha_0)$ would be a triple with $n_1 < n_0$. Finally α_0 is surjective, otherwise $\alpha_0 = \beta\gamma$ with $\gamma \in \Delta^{\mathrm{srj}}(m, n_1)$ and $n_1 < n_0$; but again the triple $(n_1, \beta(\tau_0), \gamma)$ would be a triple denying the required property of n_0. The existence of an Eilenberg triple is proved and uniqueness remains to be proved.

Let (n_1, τ_1, α_1) be another Eilenberg triple. The morphisms α_0 and α_1 are surjective and respective sections $\beta_0 \in \Delta^{\mathrm{inj}}(\underline{n_0}, \underline{m})$ and $\beta_1 \in \Delta^{\mathrm{inj}}(\underline{n_1}, \underline{m})$ can be constructed: $\alpha_0\beta_0 = \mathrm{id}$ and $\alpha_1\beta_1 = \mathrm{id}$. Then $\tau_0 = (\alpha_0\beta_0)(\tau_0) = \beta_0(\alpha_0(\tau_0)) = \beta_0(\sigma) = \beta_0(\alpha_1(\tau_1)) = (\alpha_1\beta_0)(\tau_1)$; but τ_0 is non-degenerate, so that $n_1 = \dim(\tau_1) \geq n_0 = \dim(\tau_0)$; the analogous relation holds when τ_0 and τ_1 are exchanged, so that $n_1 \leq n_0$ and the equality $n_0 = n_1$ is proved.

The relation $\tau_0 = \beta_0(\alpha_1(\tau_1))$ with τ_0 non-degenerate implies $\alpha_1\beta_0 = \mathrm{id}$, otherwise $\alpha_1\beta_0 = \gamma\delta$ with $\delta \in \Delta^{\mathrm{srj}}(\underline{n_1}, \underline{n_2})$ and $n_2 < n_1 = n_0$, but this implies τ_0 comes from $\gamma(\tau_1)$ of dimension n_2 again contradicting the non-degeneracy property of τ_0; therefore $\alpha_1\beta_0 = \mathrm{id}$ but this equality implies $\tau_0 = \tau_1$.

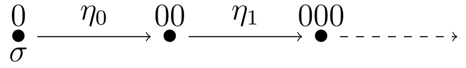
If $\alpha_0 \neq \alpha_1$, let i be an integer such that $\alpha_0(i) = j \neq \alpha_1(i)$; then the section β_0 can be chosen with $\beta_0(j) = i$; but this implies $(\alpha_1\beta_0)(j) \neq j$, so that the relation $\alpha_1\beta_0 = \mathrm{id}$ would not hold. The last required equality $\alpha_0 = \alpha_1$ is also proved. ♣

Each simplex comes from a unique non-degenerate simplex, and conversely, for any non-degenerate m-simplex $\sigma \in X_m^{ND}$, the collection $\{\alpha(\sigma)\,;\, \alpha \in \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})\,;\, n \geq m\}$ is a perfect description of all simplices coming from σ, that is, of all degenerate simplices *above* σ. This is also expressed in the following formula, describing the structure of the simplex set of any simplicial set X:

$$\coprod_{m \in \mathbb{N}} X_m = \coprod_{m \in \mathbb{N}} \coprod_{\sigma \in X_m^{ND}} \coprod_{n \geq m} \Delta^{\mathrm{srj}}(\underline{n}, \underline{m})(\sigma). \tag{12}$$

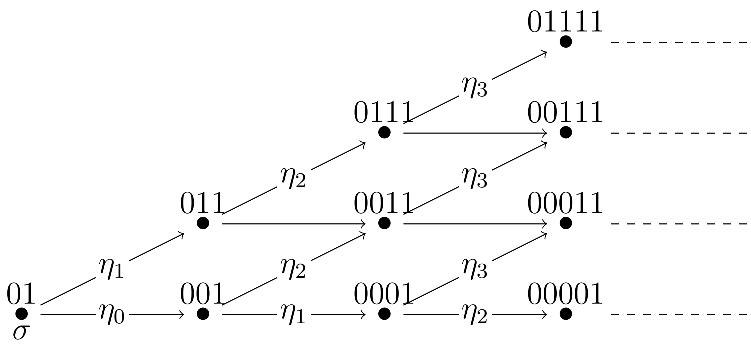
In particular a 0-simplex $v \in X_0$ is always non-degenerate, it is called a *vertex*, and such a vertex generates for every positive dimension n exactly one degenerate simplex $v_n = \eta^* v$ where η is the unique element of $\Delta^{\mathrm{srj}}(\underline{n}, \underline{0})$.

The following figures try to explain a little the nature of the collection of degenerate simplices associated to a non-degenerate simplex σ. If $\sigma \in X_0^{ND}$ is a 0-simplex, only one degeneracy in every positive dimension d, namely $\eta_{d-1} \cdots \eta_1\eta_0\sigma$. The deneneracy operator can be represented by the sequence $0 \cdots 0$, meaning the Δ-morphism $\eta_{d-1} \cdots \eta_1\eta_0 \in \underline{\Delta}(\underline{d}, \underline{0})$ sends every element of $\underline{d}$ over 0. This can be represented as follows:



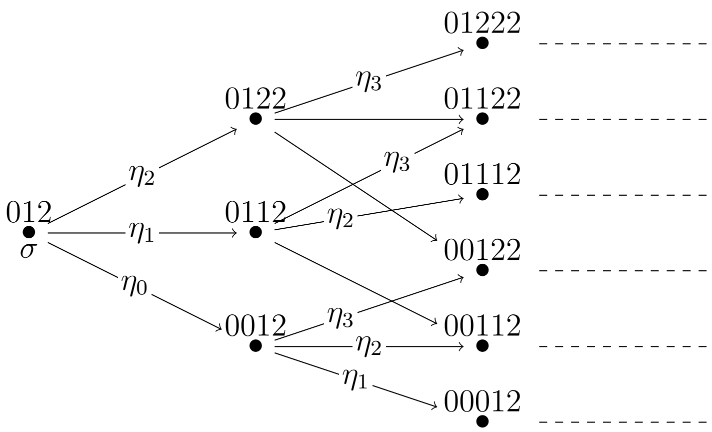
Note the expression of an arrow as *one* degeneracy operator in general is not unique. For example, the η_1 above between 00 and 000 could be replaced by η_0. But our choice directly gives the *canonical* expression $000 = \eta_1\eta_0(0)$.

If $\sigma \in X_1^{ND}$ has dimension 1, then two degeneracies in dimension 2, three in dimension 3, and so on.



On this diagram, some arrows are labeled, others not. Those which are labeled constitute a *tree* rooted at σ giving the canonical expression of a degenerate simplex from the initial one σ.

Continuing in the same way for an initial non-degenerate simplex σ of dimension 2 produces the following diagram.



with the same kind of analysis.

4 First examples.

4.1 Discrete simplicial sets.

Definition 17 — A simplicial set X is *discrete* if $X_m = X_0$ for every $m \geq 1$, and if for every $\alpha \in \Delta(\underline{m}, \underline{n})$, the induced map $\alpha^* : X_n \to X_m$ is the identity.

The reason of this definition is that the *realization* (see Section 5) of such a simplicial set is the discrete point set X_0; the Eilenberg triple of any simplex

$\sigma \in X_m = X_0$ is $(0, \sigma, \eta)$ where the map η is the unique element of $\Delta(\underline{m}, \underline{0})$; the only non-degenerate simplices are the vertices, the elements of X_0.

4.2 The simplicial complexes.

A simplicial complex $K = (V, S)$ is a pair where the first component V, the *vertex set* is an arbitrary "set"; the second component S, the *simplex set*, is made of finite subsets of V satisfying a few coherence properties, as explained in Definition 2.

The simplicial complex $K = (V, S)$ is *ordered* if the vertex set V is provided with a *total* order[1]. Then a simplicial *set*, abusively again denoted by K, is canonically associated; the simplex set of m-dimensional simplices K_m in this new framework is the set of *increasing* maps $\sigma : \underline{m} \to K$ such that the image of $\underline{m}$ is an element of S; note that such a map σ is not necessarily injective. If α is a Δ-morphism $\alpha \in \Delta(\underline{n}, \underline{m})$ and σ is an m-simplex $\sigma \in K_m$, then $\alpha(\sigma)$ is naturally defined as $\alpha(\sigma) = \sigma \circ \alpha$. A simplex $\sigma \in K_m$ is non-degenerate if and only if $\sigma \in \Delta^{\mathrm{inj}}(\underline{m}, V)$; if $\sigma \in K_m = \Delta(\underline{m}, V)$, the Eilenberg triple (n, τ, α) satisfies $\sigma = \tau \circ \alpha$ with α surjective and τ injective.

The non-degenerate m-simplices K_m^{ND} is the set of *injective* increasing maps $\underline{m} \to V$ where the image is an m-simplex of the initial simplicial *complex*. There is so a natural 1-1 correspondance between the m-simplices of the *initial* simplicial *complex* and the *non-degenerate* m-simplices of the associated simplicial *set*. The role of the degenerate simplices will be explained later.

This in particular works for $K = (\underline{d}, \mathcal{P}(\underline{d}))$ the simplex freely generated by $\underline{d}$ provided with the canonical vertex order. We obtain in this way the canonical structure of simplicial set for the *standard d-simplex* Δ^d. Its set of m-simplices Δ^d_m is the set of *increasing* maps $\Delta^d_m = \underline{\Delta}(\underline{m}, \underline{d})$; the non-degenerate simplices correspond to the injective maps in $\underline{\Delta}^{\mathrm{inj}}(\underline{m}, \underline{d})$; in particular, only one non-degenerate simplex in dimension d, namely $\mathrm{id}_{\underline{d}} \in \underline{\Delta}(\underline{d}, \underline{d})$, the *fundamental* simplex of Δ^d.

This section implies the category of simplicial complexes is essentially embedded inside the category of simplicial sets, at least if you forget this matter of order over the vertices, necessary to obtain a simplicial set. The Zermelo theorem ensures such an order over the vertex set V of the initial *complex* is always possible, but this matter of *order* plays a major role in the continuation of the story: such an order is most often *non-natural* and the consequent punishment is not far: these non-natural orders are at the origin of the role of the symmetric groups in the operadic theories. In a sense, the simplicial set theory succeeds in hiding the essential role of the symmetric groups in our geometrical space. But the revenge of the symmetric groups will be terrible: you rejected the symmetric groups at the geometrical level? Yes, but they will appear again in the algebraic framework later: under the notion of E_∞-*operad*.

The category of simplicial sets is designed to allow more flexible combinatorial construction processes than those that are possible in the the framework of

[1] Other situations where the order is not total are also interesting but will be considered later.

simplicial complexes, as roughly explained in Section 2.4.

4.3 The spheres.

Let d be a natural number. The simplest version $S = S^d$ of the d-sphere as a simplicial *set* is defined as follows: the set of m-simplices S_m is $S_m = \{*_m\} \coprod \Delta^{\text{srj}}(\underline{m}, \underline{d})$; if $\alpha \in \Delta(\underline{n}, \underline{m})$ and σ is an m-simplex $\sigma \in S_m$, then $\alpha(\sigma)$ depends on the nature of σ:

1. If $\sigma = *_m$, then $\alpha(\sigma) = *_n$;
2. Otherwise $\sigma \in \Delta^{\text{srj}}(\underline{m}, \underline{d})$ and if $\sigma \circ \alpha$ is surjective, then $\alpha(\sigma) = \sigma \circ \alpha$, else $\alpha(\sigma) = *_n$ (the emergency solution when the natural solution does not work).

This is nothing but the canonical quotient, in the simplicial *set* framework, of two simplicial *complexes* $S^d = \Delta^d/\partial\Delta^d$, at least if $d > 0$; see the figure p. 11 which illustrates how the 2-sphere can be understood as the quotient $S^2 = \Delta^2/\partial\Delta^2$. More generally the notion of simplicial subset is naturally defined and a quotient then appears. In the case of the construction of $S^d = \Delta^d/\partial\Delta^d$, the subcomplex $\partial\Delta^d$ is made of the simplices $\alpha \in \Delta(\underline{m}, \underline{d})$ that are not surjective.

The Eilenberg triple of $*_m$ is $(0, *_0, \alpha)$ where α is the unique element of $\Delta(\underline{m}, \underline{0})$. The Eilenberg triple of $\sigma \in \Delta^{\text{srj}}(\underline{m}, \underline{d}) \subset S_m$ is (d, id, σ). There are only two non-degenerate simplices, namely $*_0 \in S_0$ and $\text{id}(\underline{d}) \in S_d$, even if $d = 0$.

5 Realization.

5.1 Definition and first results.

Before giving other examples of simplicial sets, it is time now to examine the notion of *realization* in the framework of the category of simplicial sets.

Let $X = (\{X_m\}_m, \{X_\alpha\}_\alpha)$ be a simplicial set; the index m runs the non-negative integers $\mathbb{N}$; the index α runs the Δ-morphisms: a possible α is an increasing map $\alpha : \underline{m} \to \underline{n}$.

Definition 18 — The ("expensive") realization $|X|$ of X is:

$$|X| = \coprod_{m \in \mathbb{N}} X_m \times |\Delta^m| \;/\; \approx . \tag{13}$$

Each component of the coproduct is the product of the *discrete* set of m-simplices X_m by the standard *geometric* m-simplex $|\Delta^m|$, that is, the usual *topological* m-simplex; in other words, each "abstract" simplex σ in X_m gives birth to a geometric simplex $\{\sigma\} \times |\Delta^m|$, and they are attached to each other following the instructions of the equivalence relation $\approx$, to be defined. Let $\alpha \in \Delta(\underline{m}, \underline{n})$ be some

Δ-morphism, and let σ be an n-simplex $\sigma \in X_n$ and $t \in |\Delta^m| \subset \mathbb{R}^{\underline{m}}$. Then the pairs $(\alpha^*\sigma, t)$ and $(\sigma, \alpha_* t)$ are declared equivalent. Here $\alpha_* : |\Delta^m| \to |\Delta^n|$ is the (affine) *geometrical* map covariantly induced between *geometrical* simplices by the "abstract" map $\alpha : \underline{m} \to \underline{n}$ between the vertices of these simplices, according to the usual numbering. The map $\alpha^* : X_n \to X_m$ is induced by the simplicial structure: $\alpha^* = X(\alpha)$; as usual, the *sup*-$*$ intends to recall the *contravariant* nature of the association process. Frequently we omit the sub-$*$ or the sup-$*$ when the context clearly implies it.

It is not obvious to understand what is the topological space so obtained. A description a little more explicit but also a little more complicated explains more satisfactorily what should be understood.

The *cheap* realization $\|X\|$ of the simplicial set X is:

$$\|X\| = \coprod_{m \in \mathbb{N}} X_m^{ND} \times |\Delta^m| \; / \; \approx \tag{14}$$

where the equivalence relation $\approx$ is defined as follows. Let σ be a non-degenerate m-simplex and i an integer $0 \leq i \leq m$; let also $t \in |\Delta^{m-1}|$; the abstract $(m-1)$-simplex $\partial_i^* \sigma$ has a well defined Eilenberg triple (n, τ, α); then we decide to declare equivalent the pairs $(\sigma, \partial_{i*}(t)) \approx (\tau, \alpha_*(t))$.

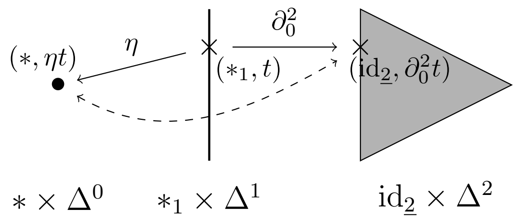
Fewer simplices are invoked in the cheap realization: only the *non*-degenerate simplices are used, but the equivalence relation assembling them to each other is more sophisticated.

For example let $S = S^d$ be the claimed simplicial version of the d-sphere described in Section 4.3. This simplical set S has only two non-degenerate simplices, one in dimension 0, the other one in dimension d. The *cheap* realization $||S||$ needs a point $|\Delta^0| = \{*\}$ and a geometric d-simplex $|\Delta^d|$ corresponding to the abstract simplex $\mathrm{id} \in \Delta(\underline{d}, \underline{d})$; then if $t \in |\Delta^{d-1}|$ and $0 \leq i \leq d$, the equivalence relation asks for the Eilenberg triple of $\partial_i(\mathrm{id}) = *_{d-1}$ which is $(0, *_0, \eta)$, the map η being the unique element of $\Delta(\underline{d-1}, \underline{0})$. Finally the initial pair $(\mathrm{id}, \partial_{i*} t)$ in the realization process must be identified with the pair $(*_0, \Delta^0)$; in other words $\|S\| = |\Delta^d| / \partial |\Delta^d|$, homeomorphic to the unit d-ball with the boundary collapsed to one point: the result is clearly a $(d-1)$-sphere.

You observe in this simple example about spheres the role of the degenerate simplices. Let us now consider the *expensive* realization $|S|$ of the simplicial set S, simplicial model of the 2-sphere S^2. In the rough description page 10 of a 2-sphere as a simplicial set, we explained we would like to attach the whole boundary $\partial|\Delta^2|$ of the triangle $|\Delta^2|$, for example its 0-face $\partial_0|\Delta^2|$ to the base point '$*$'. Two Δ-morphisms are invoked in the necessary attachment process:

- The unique map $\eta : \underline{1} \to \underline{0}$ is surjective non-injective. The *contravariant* functor which defines the simplicial set S has in particular a map $\eta^* : S_0 \to S_1$ which associates to the base point $* \in S_0$, in fact the unique element of S_0, the *degenerate* 1-simplex $*_1 \in S_1$.
- The face map $\partial_0^2 : \underline{1} \to \underline{2}$, that is, the unique injective map which avoids 0

(see page 12), applied to the 2-simplex $\mathrm{id}_{\underline{2}} \in S_2$, gives again $\partial_0^{2*}(\mathrm{id}_{\underline{2}}) = *_1$, see in Section 4.3 the Rule 2 for the simplicial description of S^n.



Now let $t \in |\Delta^1|$. In the expensive realization process, we must identify:

$$S_0 \times |\Delta^0| \ni (*, 0) = (*, \eta t) \sim (\eta^* *, t) = (*_1, t) \in S_1 \times |\Delta^1|, \quad (15)$$

and we must also identify:

$$S_2 \times |\Delta^2| \ni (\mathrm{id}_{\underline{2}}, \partial_0^2 t) \sim (\partial_0^{2*} \mathrm{id}_{\underline{2}}, t) = (*_1, t) \in S_1 \times |\Delta^1|. \quad (16)$$

Finally we may forget the point $(*_1, t)$ and directly identify $(*, 0) \sim (\mathrm{id}_{\underline{2}}, \partial_1^2 t)$. This being valid for every $t \in |\Delta^1|$, and for ∂_1^2 and ∂_2^2 as well, finally the whole boundary $\partial|\Delta^2|$ is identified to the base point $*$.

In this way, any point (σ, t) of the expensive realization, where the (abstract) simplex component σ is degenerate, can be canonically replaced by the point $(\tau, \alpha t)$ in the same realization if (n, τ, α) is the Eilenberg triple of σ, where τ is a non-degenerate simplex. The non-degenerate simplices finally *do not contribute* in the realization, but they are the necessary intermediary objects to describe the possibly sophisticated attachments.

Proposition 19 — *Both realizations, the expensive one and the cheap one, of a simplicial set X are canonically homeomorphic.*

♣ The homeomorphism $f : |X| \to \|X\|$ to be constructed maps the equivalence class of the pair $(\sigma, t) \in X_m \times \Delta^m$ to the (equivalence class of the) pair $(\tau, \alpha_*(t)) \in X_n \times \Delta^n$ if the Eilenberg triple of σ is (n, τ, α). The inverse homeomorphism g is induced by the canonical inclusion $\coprod X_m^{ND} \times \Delta^m \hookrightarrow \coprod X_m \times \Delta^m$. These maps must be proved coherent with the defining equivalence relations and inverse to each other.

If $\alpha = \beta\gamma$ is a Δ-morphism expressed as the composition of two other Δ-morphisms, then an equivalence $(\sigma, \beta_*\gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$ can be considered as a consequence of the relations $(\sigma, \beta_*\gamma_* t) \approx (\beta^*\sigma, \gamma_* t)$ and $(\beta^*\sigma, \gamma_* t) \approx (\gamma^*\beta^*\sigma, t)$, so that it is sufficient to prove the coherence of the definition of f with respect to the *elementary* Δ-operators, that is, the face and degeneracy operators.

Let us assume the Eilenberg triple of $\sigma \in X_m$ is (n, τ, α), so that $f(\sigma, t) = (\tau, \alpha_* t)$. We must in particular prove that $f(\eta_i^*\sigma, t)$ and $f(\sigma, \eta_{i*} t)$ are coherently defined. The second image is the equivalence class of $(\tau, \alpha_*\eta_{i*} t)$; the Eilenberg triple of $\eta_i^*\sigma$ is $(n, \tau, \alpha\eta_i)$ so that the first image is the equivalence class of $(\tau, (\alpha\eta_i)_* t)$ and both image representants are even equal.

Let us do now the analogous work with the face operator ∂_i instead of the degeneracy operator η_i. Two cases must be considered. If ever the composition $\alpha\partial_i \in \Delta(\underline{m-1}, \underline{n})$ is surjective, the proof is the same. The interesting case happens if $\alpha\partial_i$ is not surjective; but its image then forgets exactly one element j ($0 \leq j \leq n$) and there exists a unique surjection $\beta \in \Delta(\underline{m-1}, \underline{n-1})$ such that $\alpha\partial_i = \partial_j\beta$. The abstract simplex $\partial_j^*\tau$ gives an Eilenberg triple (n', τ', α') and the unique possible Eilenberg triple for $\partial_i^*\sigma$ is $(n', \tau', \beta\alpha')$. Then, on one hand, the f-image of $(\sigma, \partial_{i*}t)$ is $(\tau, \alpha_*\partial_{i*}t) = (\tau, \partial_{j*}\beta_*t)$; on the other hand the f-image of $(\partial_i^*\sigma, t)$ is $(\tau', \alpha_*\beta_*t)$; but according to the definition of the equivalence relation $\approx$ for $\|X\|$, both f-images are equivalent. The coherence of f is proved.

Let $\sigma \in X_m^{ND}$, $0 \leq i \leq m$, $t \in \Delta^{m-1}$ and (n, τ, α) (the Eilenberg triple of $\partial_i^*\sigma$) be the ingredients in the definition of the equivalence relation for $\|X\|$; the pairs $(\sigma, \partial_{i*}t)$ and (τ, α_*t) are declared equivalent in $\|X\|$; the map g is induced by the canonical inclusion of coproducts, so that we must prove the same pairs are also equivalent in $|X|$. But this is a transitive consequence of $(\sigma, \partial_{i*}t) \approx (\partial_i^*\sigma, t) = (\alpha^*\tau, t) \approx (\tau, \alpha_*t)$. We see here we had only described the binary relations *generating* the equivalence relation $\approx$; the defining relation is not necessarily stable under transitivity. The coherence of g is proved.

The relation $fg = \text{id}$ is obvious. The other relation $gf = \text{id}$ is a consequence of the equivalence in $|X|$ of $(\sigma, t) \approx (\tau, \alpha_*t)$ if the Eilenberg triple of σ is (n, τ, α). ♣

5.2 Simplicial model for classifying spaces.

5.2.1 The general case of a discrete group.

Illustrating the notion of realization with the classifying spaces of discrete groups is interesting. This construction can be extended to any arbitrary simplicial group, see [13, §21].

Definition 20 — Let G be a discrete group, possibly non commutative; the unit of G is denoted 1. The *classifying space* BG of G is the simplicial set defined as follows:

- The simplex set BG_m of m-dimensional simplices is $BG_m = G^m$, the elements of which are called *"m-bars"* and are traditionally denoted by $\sigma = [g_1|\cdots|g_m]$: the separator '|', a *bar*, is here preferred to the common comma for clarity.

- Face and degenerator operators are defined by:

$$\begin{aligned}
\partial_0[g_1|\cdots|g_m] &:= [g_2|\cdots|g_m];\\
\partial_m[g_1|\cdots|g_m] &:= [g_1|\cdots g_{m-1}];\\
\partial_i[g_1|\cdots|g_m] &:= [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_m];\\
\eta_i[g_1|\cdots|g_m] &:= [g_1|\cdots|g_i|1|g_{i+1}|\cdots|g_m].
\end{aligned} \tag{17}$$

In particular $BG_0 = \{[\,]\}$ has only one element.

The m-simplex $[g_1|\ldots|g_m]$ is degenerate if and only if one of the G-components is the unit element.

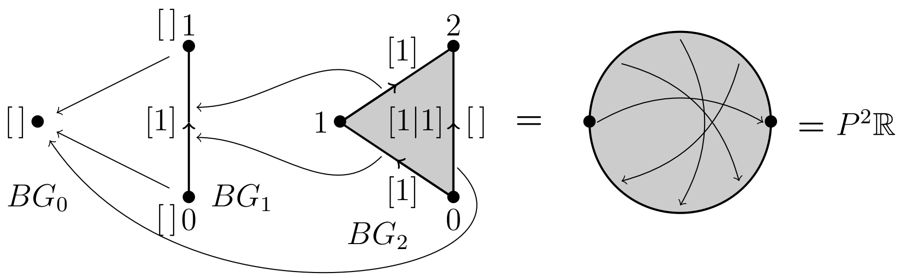
The various commuting relations must be verified; this works but does not give obvious indications on the very nature of this construction; in fact there is a more conceptual description. Let us define the simplicial set EG by $EG_m = \underline{\text{Set}}(\underline{m}, G) = G^{\underline{m}} \cong G^{m+1}$, that is, the maps of $\underline{\text{m}}$ to G without taking account of the ordered structure of $\underline{\text{m}}$ (the group G is not ordered); if $\alpha \in \Delta(\underline{n}, \underline{m})$ there is a canonical way to define $\alpha : EG_m \to EG_n$; it would be fairly coherent to write $EG = G^{\Delta}$.

There is a canonical left action of the group G on EG, and BG is the natural quotient of EG by this action. A simplex $\sigma \in EG_m$ is nothing but a $(m+1)$-tuple $(g_0, \ldots, g_m)$ and the action of g gives the simplex $(gg_0, \ldots, gg_m)$. If two simplices are G-equivalent, the products $g_{i-1}^{-1} g_i$ are the same; the quotient BG-simplex $[g_1, \ldots, g_m]$ denotes the equivalence class of all the EG-simplices $(g, gg_1, gg_1g_2, \ldots)$, which can be imagined as a simplex where the *edge* between the vertices $i-1$ and i $(i > 0)$ is labeled by g_i to be considered as a (right) operator between the adjacent vertices. Then the boundary and degeneracy operators are clearly explained and it is even not necessary to prove the commuting relations, they can be deduced of the coherent structure of EG.

5.2.2 $B\mathbb{Z}_2$ is a real projective space.

Let us examine what happens for the smallest non-trivial particular case, that is, the group G with two elements $G = \mathbb{Z}/2\mathbb{Z} =: \mathbb{Z}_2$; it is a commutative group and we then prefer to denote 0 the "unit". For a bar $[g_1|\cdots|g_m]$, two choices only for a component g_i, and the choice $g_i = 0$ implies the simplex is degenerate. So that finally exactly one non-degenerate simplex for every dimension: $G_m^{ND} = \{[1|\cdots|1]\}$.

Let us carefully examine the beginning of the construction of the realization $B\mathbb{Z}_2$. The key point is in the next figure.



Only one non-degenerate 1-simplex [1], an interval, both ends beeing identified to the unique 0-simplex [ ]: the 1-skeleton is a circle, to be understood in fact as the projective line $P^1\mathbb{R} = S^1/\mathbb{Z}_2$, that is, the circle where opposite points are identified, which is again a circle!

Only one non-degenerate 2-simplex $[1|1]$, a triangle, with the faces $\partial_0 = [1]$, $\partial_1 = [0]$ and $\partial_2 = [1]$. The faces 0 and 2 are the non-degenerate 1-simplex, and the face 1 is degenerate, therefore collapsed over the base point, the unique vertex $[\,]$; after this collapsing, there "remains" in fact only two faces for our 2-simplex, both being identified with our 1-simplex $[1]$. Examining carefully the orientations of the faces of our triangle shows finally the 2-skeleton of our realization $|B\mathbb{Z}_2|$ is the 2-dimensional real projective space.

More generally the m-skeleton is the m-dimensional real projective space $P^m\mathbb{R}$ and the total realization $|BG|$ is the inductive limit, the infinite real projective space $P^\infty\mathbb{R}$.

In the same way, $|EG|$ is the infinite real sphere S^∞ and $|BG|$ is nothing but the quotient of this sphere by the antipodal action of $\mathbb{Z}_2$.

6 Simplicial homology.

In Section 5, the strange geometric role of the *degenerate* simplices in a simplicial set has been described. It is therefore a good opportunity to introduce now the subject of *simplicial homology*, where the role, or rather the *absence* (!) of role of the degenerate simplices is also crucial.

For the most elementary notions of homological algebra, many textbooks are available. The lecture notes [20, Section 2] of another Summer School gives a careful self-contained exposition of the most elementary parts of this subject. A useful reference, for a more extended knowledge in this rich area, is [15].

6.1 Basic definitions.

Definition 21 — Let $\mathfrak{R}$ be a unitary commutative ring, called the *coefficient ring.* Let $X = (X_n, \partial_i, \eta_i)$ be a simplicial set. The $\mathfrak{R}$-*chain complex* associated to X is the object $C_*(X, \mathfrak{R}) = (C_m(X, \mathfrak{R}), d_m)_{m\in\mathbb{Z}}$ defined as follows:

- The *chain group* $C_m(X, \mathfrak{R})$ is the null group for $i < 0$, and the free $\mathfrak{R}$-module generated by the m-simplices X_m of X if $i \geq 0$.
- The *differential* $d_m : C_m(X, \mathfrak{R}) \to C_{m-1}(X, \mathfrak{R})$ is the $\mathfrak{R}$-linear map defined by:

  $$d_m(\sigma) = \sum_{i=0}^{m} (-1)^i \partial_i \sigma. \qquad (18)$$

  when $\sigma \in X_m$.

In this definition, and in general in Homological Algebra, many indices, many index sets, are omitted, and the reader is assumed to be able to deduce them from context. The beginners do not like these conscious omissions, but experience shows it is necessary if you want to avoid terribly cumbersome notations, quickly making awkward formulas and diagrams. It is even an *art* in this activity to select in every situation the right indices to be displayed, and the others to keep hidden and underlying. It is also a fruitful activity for the reader to systematically elucidate what are the missing indices, to be sure of one's understanding.

For example $(X_m, \partial_i, \eta_i)$ should in principle be displayed as:

$$(\{X_m\}_{m\in\mathbb{N}}, \{\partial_i^m\}_{m\geq 1, 0\leq i\leq m}, \{\eta_i^m\}_{i\in\mathbb{N}, 0\leq i\leq m}). \tag{19}$$

Taking account of the very definition of a simplicial set, the reader should admit there is a unique way to complete the first formula, a little elliptic, to obtain the second one, where everything is described. And the first formula is in fact so explicit, at least if you know the underlying definitions, that it is widely preferred.

In the most elementary situations, the coefficient ring $\mathfrak{R}$ is usually the integer ring $\mathfrak{R} = \mathbb{Z}$. Otherwise, some essentially *constant* coefficient ring is most often given, which allows to frequently omit the coefficient ring and to simply write $C_m(X, \mathfrak{R}) = C_m(X)$.

For example, let us consider the 1-circle $S^1 = S$ defined as in Section 4.3. Then $S_m = \{*_m\} \coprod \Delta^{\mathrm{srj}}(\underline{m}, \underline{1})$. Let us detail the chain groups $C_m(S, \mathbb{Z}) = C_m(S)$ for $m = 0, \ldots, 3$. We must firstly describe the simplices of S_0, S_1, S_2 and S_2 and their faces.

- $S_0 = \{*_0\}$, only the base point, the unique vertex of this simplicial set, no faces.
- $S_1 = \{*_1, \mathrm{id}_{\underline{1}}\}$, every face is $*_0$, no choice.
- $S_2 = \{*_2, \eta_0, \eta_1\}$, see the notations defined Section 3.1 and Proposition 3.1. For example $\partial_0(\eta_0) = \partial_1(\eta_0) = \mathrm{id}_{\underline{1}}$, but $\partial_2(\eta_0) = *_1$; this is consequence of Rule 2 in Section 4.3 and of the commuting relations page 12. We encourage the reader to compute in the same way the faces of η_1.
- $S_3 = \{*_3, \eta_0\eta_1, \eta_0\eta_2, \eta_1\eta_2\}$. For example, $\partial_0(\eta_0\eta_2) = \partial_1(\eta_0\eta_2) = \eta_1$ and $\partial_2(\eta_0\eta_2) = \partial_3(\eta_0\eta_2) = \eta_0$.

Knowing all these faces allows the user to compute the first terms of the chain complex canonically associated to this simplicial set $S = S^1$:

$$(C_0 = \mathbb{Z}) \xleftarrow{d_1} (C_1 = \mathbb{Z}^2) \xleftarrow{d_1} (C_2 = \mathbb{Z}^3) \xleftarrow{d_2} (C_3 = \mathbb{Z}^4) \tag{20}$$

where the differentials are the matrices:

$$d_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{21}$$

We observe the composition of two successive differentials are null, this is always the case.

Proposition 22 — *If (C_n, d) is the chain complex associated to a simplicial complex X, the composition of two successive differentials $d_q d_{q+1}$ is null. This allows to define:*

- $Z_q(X, \mathfrak{R}) := \ker d_q$ *is the group of* q-cycles *of* X.
- $B_q(X, \mathfrak{R}) := im\, d_{q+1}$ *is the group of* q-boundaries *of* X.
- *The relation $d_q d_{q+1} = 0$, always satisfied, is equivalent to $B_q \subset Z_q$.*
- *The quotient group $H_q(X, \mathfrak{R}) := Z_q/B_q$ is the* q-dimensional homology group with coefficients in $\mathfrak{R}$. ♣

In the example of $C_*(S)$, we can determine:

- $Z_0 = \mathbb{Z}$, $B_0 = 0$ and $H_0 = \mathbb{Z}$.
- $Z_1 = \mathbb{Z}^2$, $B_1 = \mathbb{Z}$ direct summand of Z_1 and $H_1 = \mathbb{Z}$.
- $Z_2 = \mathbb{Z}^2$ generated by $*_2 - \eta_0$ and $*_2 - \eta_1$, so that $B_2 = Z_2$ and $H_2 = 0$.

Writing for example Z_0 '=' $\mathbb{Z}$ is not correct, in fact the cycle group Z_0 is *isomorphic* to $\mathbb{Z}$ and *equal* only to $\mathbb{Z}*_0$, the free $\mathbb{Z}$-module generated by the unique 0-simplex $*_0$. These shorthands are common, often convenient, but can also be the source of serious drawbacks when Algebraic Topology is examined from a *constructive* point of view, see [20].

Many degenerate simplices in a simplicial set! Except for examples as simple as our circle S^1, it is not easy in general to compute these homology groups. In fact, from the *homological* point of view, the role of these degenerate simplices is void! The key point is the following: in general a face of a degenerate simplex can be non-degenerate, for example above $\partial_0^2 \eta_0 = \mathrm{id}_{\underline{1}}$, but the *differential*, that is, the alternate sum of faces, of a degenerate simplex is always a combination of degenerate simplices, for example $d\eta_0 = *_1$. So that we can denote by $C_*^D(X)$ the sub-chain complex generated by the degenerate simplices. It happens this chain complex is "without" homology, which, by a difference process, produces the next proposition.

Proposition 23 — *Let X be a simplicial set, $C_*(X)$ the associated chain complex, $C_*^D(X)$ the degenerate sub-complex and $C_*^{ND}(X) := C_*(X)/C_*^{ND}(X)$ the quotient chain complex. Then the canonical projection $C_*(X) \to C_*^{ND}(X)$ induces an isomorphism between the homology groups.*

♣ [15, VIII.6]. ♣

Definition 24 — A *chain complex morphism* $f : C_* \to C'_*$ is a collection of linear maps $f = \{f_n : C_n \to C'_n\}$ compatible with the differentials: $df = fd$, that is, for every n, the relation $d_n f_n = f_{n-1} d_n$ holds.

One then says f is *of degree* 0, for f respects the degree. It is also possible to consider also maps of arbitrary degrees, but be careful in this case with sign coherences! Most often, the index n for a component f_n of a chain complex morphism is omitted, and except particular cases, elliptic formulas such as $df = fd$ are preferred.

Because of the compatibility with differentials, a chain complex morphism $f : C_* \to C'_*$ induces many natural maps, most often denoted by the same symbol f:

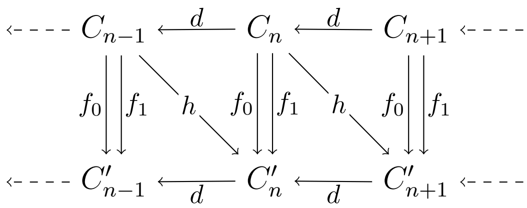
$$\begin{array}{l} f : Z_*(C_*) \to Z_*(C'_*) \\ f : B_*(C_*) \to B_*(C'_*) \\ f : H_*(C_*) \to H_*(C'_*) \end{array} \qquad (22)$$

In fact, because of the relation $df = fd$, the image of a cycle is a cycle, the image of a boundary is a boundary, so that f naturally induces a map between homology classes.

6.2 Homotopy and homology for simplicial complexes.

We will examine later in details the notion of *combinatorial homotopy* in the framework of simplicial sets, not so easy. Considering the particular case of simplicial *complexes* is a good introduction. Firstly, a *purely algebraic* notion of homotopy.

Definition 25 — Let C_* and C'_* be two chain complexes, and $f_0, f_1 : C_* \to C'_*$ two chain complex morphisms. These morphisms are (algebraically) *homotopic* if there exists an operator $h = \{h_n : C_n \to C'_{n+1}\}_{n\in\mathbb{Z}}$ satisfying $f_1 - f_0 = dh + hd$.



Our homotopy operator h has *degree* +1. If compatible with the differentials, the relation $dh = -hd$ would be satisfied[2]; our homotopy operator is in fact not at all compatible with differentials, the "error" being just the difference $dh + hd = f_1 - f_0$.

Most topologists say such maps f_0 and f_1 are *chain equivalent*; we prefer the more coherent terminology of our definition: as illustrated later, this definition is nothing but the *algebraic* translation in the chain complex framework of the topological notion of homotopy. With a warning: we will see two maps between various sorts of topological spaces which are (topologically) homotopic induce maps *algebraically* homotopic between chain complexes, but *the converse in general is false.*

[2]Not $dh = hd$, for the "right" sign is given by the famous Koszul "rule": $dh = (-1)^{\deg(d)\cdot\deg(h)} hd$.

Proposition 26 — *Let* $f_0, f_1 : C_* \to C'_*$ *be two* homotopic *chain complex morphisms. Then the induced maps* $f_0, f_1 : H_*(C_*) \to H_*(C'_*)$ *are* equal.

♣ Let $h \in H_*(C_*)$ be a homology class represented by some cycle $z \in Z_*(C_*)$. Then $(f_1 - f_0)(h)$ is represented by $(f_1 - f_0)(z) = (dh + hd)(z) = (dh)(z)$, for z cycle means $dz = 0$. The diffference cycle $f_1(z) - f_0(z)$ therefore is the *boundary* $dh(z)$ and these cycles are homologous; in other words the homology classes $f_0(h)$ and $f_1(h)$ are *equal*. ♣

Proposition 27 — *Let* K *and* K' *be two simplicial* complexes*, and* $f_0, f_1 : K \to K'$ *two simplicial morphisms which are (topologically)* homotopic*: see Definition 5 and the following discussion. Then the induced maps* $f_0, f_1 : C_*(K) \to C_*(K')$ *are (algebraically) homotopic and therefore the induced maps between homology groups* $f_0, f_1 : H_*(K) \to H_*(K')$ *are* equal*.*

This is a powerful tool for negative results: conversely, if the induced maps between homology groups are *different*, then the original continuous maps are *not* homotopic. This proposition proved here only in the simplicial complex framework in fact has a very general scope, and is at the very definition of *Algebraic* Topology: a purely algebraic observation implies topological properties.

♣ Given the hypotheses about K, K', f_0 and f_1, we have to construct an algebraic homotopy operator $h : C_*(K) \to C_{*+1}(K')$ between both chain complex morphisms f_0 and f_1. The answer is the following:

$$h((v_0, \ldots, v_n)) := \sum_{i=0}^{n} (-1)^i (f_0 v_0, \ldots, f_0 v_i, f_1 v_i, \ldots, f_1 v_n)). \tag{23}$$

To save some paper and produce less CO_2, we verify the homotopy property only in the case $n = 1$:

$$\begin{array}{rcl} h((v_0, v_1)) & = & (f_0 v_0, f_1 v_0, f_1 v_1) - (f_0 v_0, f_0 v_1, f_1 v_1); \\ dh((v_0, v_1)) & = & \quad (f_1 v_0, f_1 v_1)_1 - (f_0 v_0, f_1 v_1)_2 + (f_0 v_0, f_1 v_0)_3 \\ & & -(f_0 v_1, f_1 v_1)_4 + (f_0 v_0, f_1 v_1)_2 - (f_0 v_0, f_0 v_1)_5; \\ d((v_0, v_1)) & = & (v_1) - (v_0); \\ hd((v_0, v_1)) & = & (f_0 v_1, f_1 v_1)_4 - (f_0 v_0, f_1 v_0)_3; \\ (f_1 - f_0)((v_0, v_1)) & = & (f_1 v_0, f_1 v_1)_1 - (f_0 v_0, f_0 v_1)_5. \end{array} \tag{24}$$

where the indices after the simplex expressions show the correspondances which prove the relation $f_1 - f_0 = dh + hd$ in this particular case. The general proof is analogous, a good exercise about index handling. ♣

Is the reader really satisfied with this "proof"? He should not! We have accumulated here a terrible number of "imprecisions", let us be simple, a terrible number of *faults*. Each one is interesting and illustrates the role of the Algebraic Topology's Devil, namely the *symmetric group*.

The chain complex $C_*(K)$ associated to the simplicial complex K is given in Definition 21, which needs in turn the explanations of Section 4.2, where the simplicial *set* associated to a simplicial *complex* is defined. But in the initial Definition 1 of a simplicial complex, no order over the vertices; on the contrary, when defining "the" associated simplicial set, a total order over the vertices is required. If we change this order, what happens for example for the resulting homology groups? Yes, they are the same up to isomorphism, but the proof is not so easy.

Second difficulty, Definition 3 for a simplicial map between simplicial *complexes* does not require any compatibility conditions with vertex orders, in fact not yet considered before this definition. But after defining some orders over the vertices of K and K', if $f : K \to K'$ is a simplicial map, it can happen $v_0 < v_1$ in K and $fv_0 > fv_1$ in K', and the "induced" maps between chain groups, where the generators are made of "ordered" simplices, is then erroneously defined. Another difficulty occurs when $fv_0 = fv_1$: the image simplex is then degenerate, but we did not even mention if we preferred the total version $C_*(K)$ or the normalized one $C_*^{ND}(K)$ for "the" chain complex associated to K.

We could require the simplicial maps compatible with orders, which is very restrictive; but even with such a restriction, the mixed term:

$$(f_0 v_0, \dots, f_0 v_i, f_1 v_i, \dots, f_1 v_n)$$

of the formula (23) defining the homotopy operator h can produce a simplex with vertices in a wrong order. You see this question of orders over the simplex vertices is rather tough.

A really *complete* solution about this problem of simplex vertices can be found in [7, Chapter VI], where *two* chain complexes are associated to a simplicial complex, a big one called the *ordered* chain complex and a smaller one called the *alternating* chain complex; the last one is isomorphic to the *normalized* chain complex $C_*^{ND}(K)$ defined here through the intermediary notion of simplicial *set*; the first one accepts as generators simplices described as sequences of vertices in any order, taking account of this order; it accepts as well degenerate simplices with repetitions in the vertices. Both chain complexes have advantages and drawbacks, but their homology groups are canonically isomorphic, a frequent situation in Algebraic Topology.

7 Homology groups: a quick survey.

In this section, we give a rough "cultural" presentation of the simplest tools allowing one to compute homology groups. It is just a presentation, the results are most often stated without any demonstration and references, at least when they are reachable through any common textbook of Algebraic Topology.

You are interested in some space X, some coefficient group $\mathfrak{R}$ is given and for some reason, you would like to determine the groups $H_*(X; \mathfrak{R})$. The space X can be described as a simplicial complex, or more generally as a simplicial set, and you could consider the *simplicial* homology groups. Still more generally, the *singular* homology could be used, which is defined for arbitrary topological spaces; if the space can be *triangulated*, that is, if it is homeomorphic to some simplicial set, then the simplicial and singular homology groups are canonically isomorphic; in particular the simplicial homology groups do not depend on the chosen triangulation.

7.1 Homotopy Types.

Definition 28 — A map $f : X \to Y$ between two simplicial sets (or more generally between two topological spaces) is a *homotopy equivalence* if there exists a *homotopical inverse* $g : Y \to X$, that is, a map satisfying: gf is homotopic to id_X and fg is homotopic to id_Y. If so, it is said both spaces X and Y have the same *homotopy type*. A *homotopy type* is an equivalence class for this equivalence relation.

Proposition 27 implies the induced maps between homology groups are then *isomorphisms*. So that if you observe H_nX and H_nY are not isomorphic for some integer n, you have proved the spaces X and Y do not have the same homotopy type.

It is well known the homology groups in general do not suffice to distinguish homotopy types. The standard example is $X = S^2 \vee S^4$ and $Y = P^2\mathbb{C}$; their homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$ and the others are null ; however their homotopy types are different, which in this case is proved by considering the multiplicative structure in cohomology. This extra information is not enough in general: the next example is $X = S^3 \vee S^5$ and $Y = \Sigma P^2\mathbb{C}$, this time distinguished byt a Steenrod operation. The problem of giving a *complete* invariant set for the homotopy types is today *open* [19].

Definition 29 — A space X is *contractible* if it has the homotopy type of a point.

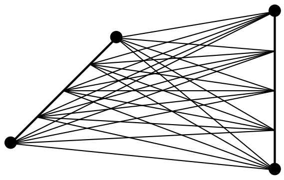
If a space X is contractible, it has the same homology groups as a point, that is, $H_0(X, \mathfrak{R}) = \mathfrak{R}$ and $H_n(X, \mathfrak{R}) = 0$ for $n > 0$. For example a simplex Δ^n is contractible. The converse is false: some non trivial discrete groups G are *acyclic*, the same homology as a point, so that the corresponding classifying space BG, see Section 5.2.1, is not contractible but with trivial homology. In the particular case of a simply connected space, then the equivalence between contractibility and trivial homology is true.

Definition 30 — Let $K = (V, S)$ and $K' = (V', S')$ be two *disjoint* simplicial complexes, that is, $V \cap V' = \emptyset$. Then the *join* $K'' = K \boxtimes K'$ is defined as $K'' = (V'', S'')$ with $V'' = V \cup V'$ and $\sigma'' \in S''$ if and only if $\sigma'' \cap V \in S \cup \{\emptyset\}$ and $\sigma'' \cap V' \in S' \cup \{\emptyset\}$, but $\sigma'' = \emptyset$ of course remains excluded[3]. In particular the *cone*

[3]It would be convenient to decide there is always, in any simplicial complex, a unique simplex of dimension -1 corresponding to the void set of vertices; it would be a sort of *augmented* simplicial complex.

CK of a simplicial complex $K = (V, S)$ is the join $CK = * \boxtimes K$ where $*$ is a simplicial complex reduced to one point, the unique vertex, this vertex not being a vertex of K.

This definition means the simplices of $K \boxtimes K'$ are made of the simplices of K, the simplices of K', and any pair (σ, σ') of $S \times S'$ generates a simplex of K'' of dimension $\dim \sigma + \dim \sigma' + 1$. For example the join of two intervals is a tetrahedron: think you have *joined* any point of the first interval to any point of the second one, which explains the terminology.



The topological definition of the join is:

$$X \boxtimes Y := (X \times [0, 1] \times Y)/\sim \tag{25}$$

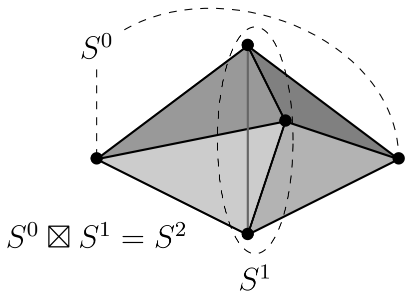
where the equivalence relation $\sim$ identifies $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$ for every $x, x' \in X$ and $y, y' \in Y$. In particular, if X has only one point, we find only $* \boxtimes Y = (I \times Y)/(\{0\} \times Y) = CY$.

To construct a cone CK for a simplicial complex K, you just have to add to every simplex σ of K the simplex $\{*\} \cup \sigma$. For example, the cone of an n-simplex is an $(n + 1)$-simplex.

A non-trivial exercise consists in proving the join of two spheres is a sphere :

$$S^p \boxtimes S^q \cong S^{p+q+1}. \tag{26}$$

Here you should take the simplicial definition $S^p = \partial\Delta^{p+1}$ and the same for S^q. The resulting simplicial complex is not isomorphic to ∂S^{p+q+1}, but its realization is homeomorphic to. Hint: In a Euclidian S^{p+q+1} sphere, you have two "orthogonal" disjoint "large" spheres S^p and S^q; for example in the ordinary 2-sphere, you have two remarkable spheres S^0 and S^1 which are orthogonal: S^0 could be made of both North and South poles, S^1 being the equator; many other solutions with the same geometry.



A cone CK is always contractible, so that the homology groups of a cone are canonically isomorphic to the homology groups of a point.

7.2 Exact sequences.

Definition 31 — An *exact sequence* is a linear diagram of groups and group morphisms:

$$\cdots \longleftarrow F \xleftarrow{f} G \xleftarrow{g} H \longleftarrow \cdots \tag{27}$$

where the kernel of every arrow is the image of the previous one. For example, in the displayed case, $\ker(f) = \operatorname{im}(g)$. The inclusion $\operatorname{im}(g) \subset \ker(f)$ is equivalent to $fg = 0$, that is, the sequence is a chain complex. Asking for the equality $\ker(f) = \operatorname{im}(g)$ is claiming the corresponding homology group is null: an exact sequence is a chain complex the homology groups of which are null.

In particular a *short exact sequence* is a diagram:

$$0 \longleftarrow F \xleftarrow{f} G \xleftarrow{g} H \longleftarrow 0 \tag{28}$$

where f is *surjective*, $\ker(f) = \operatorname{im}(g)$ and g is *injective.*

Frequently, results about homology groups are presented as exact sequences. An important particular case of this sort is the Mayer-Vietoris exact sequence.

Theorem 32 (Mayer-Vietoris exact sequence) — *Let X be a simplicial set, A and B two simplicial subsets such as $X = A \cup B$. Then there exists a canonical long exact sequence:*

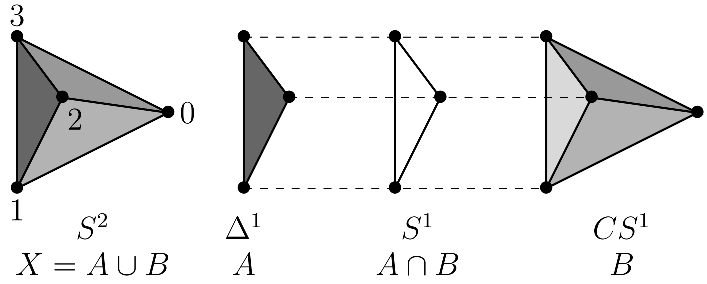
$$\cdots \longleftarrow H_{n-1}(A \cap B) \xleftarrow{\partial} H_n(X) \xleftarrow{j_A \oplus j_B} H_n(A) \oplus H_n B \xleftarrow{i_A \oplus (-i_B)} \cdots$$
$$\cdots \xleftarrow{i_A \oplus (-i_B)} H_n(A \cap B) \xleftarrow{\partial} H_{n+1}(X) \longleftarrow \cdots$$

The maps i_A, i_B, j_A and j_B are induced by the canonical inclusions $i_A : A \cap B \hookrightarrow A$, $i_B : A \cap B \hookrightarrow B$, $j_A : A \hookrightarrow X$ and $j_B : B \hookrightarrow X$. Note the minus sign given to i_B, necessary to obtain $(j_A \oplus j_B) \circ (i_A \oplus (-i_B)) = 0$. The maps $\partial : H_n(X) \to H_{n-1}(A \cap B)$ are the *connection* morphisms, more esoteric, not defined here.

The Mayer-Vietoris exact sequence is important: it allows you to have informations about the groups $H_*(X)$ when you know the homology groups $H_*(A)$, $H_*(B)$ and $H_*(A \cap B)$: in many cases you can so deduce the homology groups of the total space X when you know the homology groups of three of its constituents, A, B and $A \cap B$.

As a typical example, let us assume n is an integer $n \geq 2$.

Proposition 33 — *There exists a canonical isomorphism $H_p(S^n) \cong H_{p-1}(S^{n-1})$ for $p \geq 2$.*



In this figure illustrating the particular case $n = 2$, the 2-sphere S^2 is the *boundary* of the 3-simplex; two components in the decomposition, the "lid" $A = \Delta^1 = \{1, 2, 3\}$ and the "cornet" cone CS^1 of the circle S^1, the boundary of the lid, with respect to the vertex 0.

♣ Let us consider the n-sphere S^n as the boundary of the $(n+1)$-simplex, that is the simplex spanned by $\underline{n+1} = \{0 \ldots n+1\}$. In particular the $(n-1)$-sphere S^{n-1} is the boundary of the n-simplex A spanned by $\{1 \ldots n+1\}$. We can also consider the simplicial subcomplex B defined as the cone of S^{n-1} of summit 0. Then $A \cap B = S^{n-1}$ and A and B are contractible. Let us consider the following segment of the Mayer-Vietoris exact sequence:

$$H_{p-1}(A) \oplus H_{p-1}(B) \longleftarrow H_{p-1}(A \cap B) \longleftarrow H_p(X) \longleftarrow H_p(A) \oplus H_p(B) \qquad (29)$$

which becomes:

$$0 \longleftarrow H_{p-1}(S^{n-1}) \longleftarrow H_p(S^n) \longleftarrow 0 \qquad (30)$$

for A and B are contractible. This exact sequence implies the central map is an isomorphism. ♣

Examining carefully in the same way the beginning of the Mayer-Vietoris exact sequence, and taking account of the easy calculation of $H_*(S^1)$, we obtain for $n \geq 1$:

$$\begin{array}{rcl} H_0(S^n; \mathfrak{R}) & = & \mathfrak{R}, \\ H_n(S^n; \mathfrak{R}) & = & \mathfrak{R}, \\ H_p(S^n; \mathfrak{R}) & = & 0 \text{ otherwise.} \end{array} \qquad (31)$$

7.3 About the coefficient group.

According to the problem, some or other coefficient group $\mathfrak{R}$ is preferred. Frequently it is $\mathbb{Z}$ the integer ring; this coefficient group can also be a *field*, in particular a field $\mathbb{Z}/p\mathbb{Z}$ for a prime p, or also the field $\mathbb{Q}$ of the rational numbers, the field $\mathbb{R}$ of the real numbers. The case where the coefficient group is a field $\mathbb{F}$ is often easier, for when a short exact sequence $0 \to \mathbb{F}^a \to H \to \mathbb{F}^b \to 0$ is used to determine the unknown group H, then the group H is isomorphic to the direct sum $\mathbb{F}^a \oplus \mathbb{F}^b = \mathbb{F}^{a+b}$: no *extension problem* in this case to determine the isomorphism class of H. On the contrary, if the coefficient group is $\mathbb{Z}$, then an exact sequence such as $0 \to \mathbb{Z}/6\mathbb{Z} \to H \to \mathbb{Z}/2\mathbb{Z} \to 0$ leaves the user of this exact sequence with a doubt, for the group H could be either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$: determining the right choice needs further informations about which most books about Algebraic

Topology are not very loquacious. Obtaining automatic algorithms (pleonasm) efficiently solving this problem is relatively recent [18].

The coefficient group $\mathbb{Z}$ is *universal*, which needs the notion of torsion group to be described.

Definition 34 — If G and H are commutative groups, let $0 \to G_1 \to G_0 \to G \to 0$ *be a* free presentation *of the group G as $\mathbb{Z}$-module. Then the tensor product morphism $(G_1 \to G_0) \otimes H$ is not necessarily injective and its kernel is called the torsion group* $\mathrm{Tor}_{\mathbb{Z}}(G, H)$.

You must take for G_0 an arbitrary *free* $\mathbb{Z}$-module allowing you to define a surjection $G_0 \to G$; you could take for example the free $\mathbb{Z}$-module *generated* by G itself: $G_0 = \mathbb{Z}^{(G)}$ and the map sending the generator g of G_0 over the element g of G. The kernel of the surjection $G_0 \to G$ is necessarily free, and it is the group G_1. Frequently, more simple choices for G_0 are possible, and the final result *will not depend* on the choice of the resolution. There is also a canonical isomorphism $\mathrm{Tor}(G, H) \cong \mathrm{Tor}(H, G)$: the procedure can be applied to H if you prefer.

The simplest toy example is $G = \mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{Z}/4\mathbb{Z}$. The cheapest resolution of $\mathbb{Z}/2\mathbb{Z}$ is:

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{32}$$

The tensor product by $\mathbb{Z}/4\mathbb{Z}$ produces:

$$0 \to \mathbb{Z}/4\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{33}$$

where the map $\mathbb{Z}/4\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}/4\mathbb{Z}$ is *not injective*, producing the group:

$$\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) := \ker(\mathbb{Z}/4\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}. \tag{34}$$

The reader is advised to verify $\mathrm{Tor}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ gives the same result.

Theorem 35 — *If X is a space, and $\mathfrak{R}$ some coefficient group, a short exact sequence is canonically defined:*

$$0 \leftarrow \mathrm{Tor}(H_{n-1}X, \mathfrak{R}) \leftarrow H_n(X; \mathfrak{R}) \leftarrow H_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathfrak{R} \leftarrow 0 \tag{35}$$

Furthermore this exact sequence is split*, meaning the solution of the extension problem is necessarily $H_n(X; \mathfrak{R}) \cong (H_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathfrak{R}) \oplus \mathrm{Tor}(H_{n-1}X, \mathfrak{R})$, but the splitting is not canonical.*

This theorem implies the knowledge of the groups $H_*(X; \mathbb{Z})$ is enough to determine the other homology groups $H_*(X; \mathfrak{R})$. These homology groups with integer coefficients are therefore so important that most often they are simply denoted by H_*X.

Conversely, the cost of determining these groups is high, not amazing: the Universal Coefficients Theorem shows the $\mathbb{Z}$-groups contain all the others. In the case, frequent, where these homology groups are known as $\mathbb{Z}$-modules of finite type, it can be on the contrary more efficient to subdivide the $\mathbb{Z}$-problem into subproblems as follows:

$$H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{d_0} \oplus \bigoplus_{p \text{ prime}} H_n(X;\mathbb{Z}_{(p)}). \tag{36}$$

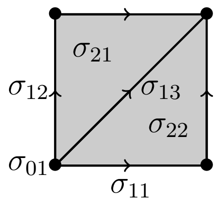
where d_0 is the $\mathbb{Q}$-dimension of $H_n(X;\mathbb{Q})$ and $\mathbb{Z}_{(p)}$ is the ring integer $\mathbb{Z}$ localized at the prime p. And *specific methods* with respect to $\mathbb{Q}$ and $\mathbb{Z}_{(p)}$ often allow to compute these homology groups.

7.4 Elementary computations.

If the space X is described as a *finite* simplicial complex, or a finite simplicial *set*, then the computation of its homology groups is *in principle* elementary. You have to compute the boundary matrices $d_n : C_n(X) \to C_{n-1}(X)$ and $d_{n+1} : C_{n+1}(X) \to C_n(X)$ and an elementary computation of $\mathbb{Z}$-linear algebra, mainly the Smith reduction of integer matrices, produces the quotient $\ker d_n / \operatorname{im} d_{n+1}$. But the practical computations become quickly painful or even impossible, even with the most powerful computers.

For example a triangulation of the torus $T_2 = S^1 \times S^1$ is given p.60 with 9 vertices, 27 edges and 18 triangles. Computing naively $H_1(T_2)$ needs the matrix describing d_1, 27 columns and 9 rows, and the matrix describing d_2, 18 columns and 27 rows. With pencil and paper, it is necessary you are a little bit lucid and careful: you should determine the matrix d_1 has rank 8, producing an image of rank 8 and a kernel of rank 19; the matrix d_2 has rank 17, image of rank 17 and kernel of rank 1. A little more work will produce $H_1 = \mathbb{Z}^2$.

The benefit of the technology of simplicial *sets* becomes obvious if you consider the following triangulation of the torus:



As before you must identify on the one hand the left and right edges σ_{12} of the square, and on the other hand the top and bottom edges σ_{11}, and the four corners σ_{01} are so identified, producing the unique vertex of this simplicial set. Finally, 1 vertex σ_{01}, 3 edges σ_{11}, σ_{12} and σ_{13}, and two triangles σ_{21} and σ_{22}. The boundary matrices then are:

$$d_1 = [0\ 0\ 0]\,, \quad d_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \tag{37}$$

It is then significantly easier to deduce the homology groups, the same as before, for the isomorphism classes of these groups do not depend on the chosen triangulation, as a simplicial complex or as a simplicial set as well.

But even with this technology of simplicial sets, you could meet severe difficulties with these so called "elementary" computations. For example the standard triangulation of the Eilenberg-MacLane $K(\mathbb{Z}/2\mathbb{Z}, 4)$ (see Section 9.1) requires n_d simplices of dimension d with in particular:

$$\begin{aligned} n_7 &= 34359509614 \\ n_8 &= 1180591620442534312297 \\ n_9 &= 85070591730234605240519066638188154620 \end{aligned} \tag{38}$$

and no computer in this planet can store into its memory the boundary matrices d_8 and d_9 if you intend to compute "elementarily" the group $H_8K(\mathbb{Z}/2\mathbb{Z}, 4)$. Fortunately, more powerful methods, mainly the Eilenberg-Moore spectral sequence, allow the interested algebraic topologist to reduce this computation to the homology group of another chain complex, where this time, $n'_7 = 4$, $n'_8 = 8$ and $n'_9 = 15$, more reasonable; the nature of the corresponding chain complex is much more sophisticated, the differentials of the generators are not easy to determine, but once it is done, the matrices can be easily implemented and the homology group quickly computed. To be complete about this subject of the homology groups of the terrible Eilenberg-MacLane spaces, we must signal the nice work of... Eilenberg and MacLane (see in particular [5, 6]) finally led Henri Cartan [3] to find a very elegant method directly giving the homology groups of these Eilenberg-MacLane spaces.

7.5 Find simpler spaces!

Most often, computing $H_*(X)$ amounts to expressing X as the result of a construction, as elementary as possible, from simpler spaces, the homology of which being known.

7.5.1 Amalgamated sums.

The Mayer-Vietoris Theorem 32 is a typical example, where X is expressed as the union of two components A and B, the homology of which being known, and also the homology of the intersection $A \cap B$. This is to be considered as an amalgamated sum: X is nothing but the sum of A and B "amalgamated" along their intersection $A \cap B$.

7.5.2 Products.

Speaking of *sums* of spaces naturally leads to think of *products*. The next section explains how to define the product of simplicial sets, a simplicial version of the ordinary product of topological spaces. Let us assume the homology groups of X and Y are known, how to determine the homology groups of $X \times Y$? The answer is the Künneth theorem.

Theorem 36 (Künneth Theorem) — *Let X and Y be two spaces and $Z = X \times Y$ their product. Then a canonical short exact sequence is defined for every integer n:*

$$0 \longleftarrow \bigoplus_{i=0}^{n-1} \mathrm{Tor}(H_i(X), H_{n-i-1}(Y)) \longleftarrow H_n(X \times Y) \longleftarrow \cdots$$
$$\cdots \longleftarrow H_n(X \times Y) \longleftarrow \bigoplus_{i=0}^{n} (H_i(X) \otimes H_{n-i}(Y)) \longleftarrow 0$$

Some coefficient group $\mathfrak{R}$ is underlying, all the homology groups are computed with respect to this coefficient group, the same for the tensor products and torsion products. This exact sequence is split, but the splitting is not canonical.

This Künneth formula is to be compared with the product of polynomials or power series:

$$(\sum_n a_n X^n) \times (\sum_n b_n X^n) = \sum_n (\sum_{i=0}^{n} a_i b_{n-i}) X^n \tag{39}$$

which suggests the last term of the Künneth exact sequence. This comparison between product properties leads to the interesting and powerful notion of *Poincaré series*. A “slight” error is generated by the naive product, expressed through torsion products in a similar way, with one dimension less, like in the Universal Coefficients Theorem. In fact the Universal Coefficients Theorem is a consequence of a more general statement of the Künneth theorem in the framework of arbitrary chain complexes.

In the particular case where the coefficient group $\mathfrak{R}$ is a field, or also when one of the factors has its homology groups free, which implies the torsion groups are null, then the Künneth formula quickly gives the homology groups of the product. For example, the formulas (31) give the homology groups of the n-sphere S^n. The Künneth formula then implies for the product $X = S^m \times S^n$ when $1 \leq m < n$:

$$\begin{array}{l} H_0 = H_m = H_n = H_{m+n} = \mathfrak{R}; \\ H_k = 0 \ \text{ if } k \neq 0, m, n, m+n. \end{array} \tag{40}$$

and the case $m = n$ can be processed in the same way.

The simplest case where torsion products are involved is the case of $X = P^2\mathbb{R} \times P^2\mathbb{R}$. The *integer* homology groups of $P^2\mathbb{R}$ are $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/2\mathbb{Z}$ and the others are null. The Künneth formula then produces $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_2(X) = \mathbb{Z}/2\mathbb{Z}$ and finally $H_3(X) = \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

7.5.3 Twisted products ⇒ Serre spectral sequence.

It is explained Section 12 how a product of simplicial sets can be *twisted* to obtain a much larger variety of spaces. It is the simplicial version of the notion of *fibration*. The general framework is as follows, you have to consider a diagram or spaces which is similar to a short exact sequence:

$$* \rightarrow F \hookrightarrow E \rightarrow B \rightarrow * \tag{41}$$

Most often, the initial and terminal point spaces are omitted:

$$F \hookrightarrow E \to B \tag{42}$$

but they should not, for the analogy with the short exact sequences is really good. The first component F is called the fiber space, the last one B is the base space, and the central space E is the total space. The simplest case happens when this "exact sequence" is split; you must consider B as a pointed space, a point $* \in B$ is given. Then a particular case of twisted product, in fact in this case not twisted, is the following:

$$F \hookrightarrow F \times B \to B \tag{43}$$

where the first arrow is the map $x \mapsto (x, *)$, this is why a base point in B is required, and the second arrow is the canonical projection. The general situation could be written:

$$F \hookrightarrow F \times_\tau B \to B \tag{44}$$

where the index τ means some method is used to change the trivial product $F \times B$ into a deeply modified one $F \times_\tau B$. The simplicial case is described in detail in Section 12.

The simplest non-trivial case maybe is the trivial product $\mathbb{Z} \times S^1$ producing a space with a countable number of circles, which can be modified to $\mathbb{Z} \times_\tau S^1 = \mathbb{R}$ giving this "exact sequence":

$$\{0\} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{R} \to S^1 \to * \tag{45}$$

with in particular the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the exponential map $\exp : \mathbb{R} \to S^1 : t \mapsto e^{2i\pi t}$, considering S^1 as the unit circle of the complex plane $\mathbb{C}$. A detailed description of this particular case is given Section 12.1; note also this topological twisted product is simultaneously an actual exact sequence of *groups*, where the central group is a non-trivial extension of the last one S^1 by the first one $\mathbb{Z}$. In more general situations, this is nomore true.

What about the homology groups of a twisted product? The homology groups $H_*(F)$ of the fiber space F and the homology groups $H_*(B)$ of the base space B are assumed known and you intend to compute the homology groups of the total space $E = F \times_\tau B$. How to process?

We have explained when commenting the Künneth Theorem 36 that the homology groups of a product can be more or less interpreted as the product of the homology groups of the factors, if you consider the "total" homology H_* as a Poincaré series $\sum H_n t^n$. For the twisted product, there remains in a sense to twist again the Künneth result, to be considered in this situation as an intermediary step. This complex twisting process for the homology groups is formalized through the important notion of *spectral sequence*. But this process works only if the base space is *simply connected*, and therefore cannot be applied to our example of the exponential map $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$: the first homotopy group $\pi_1 S^1 = \mathbb{Z}$ is non-trivial and the circle S^1 is certainly the simplest example of a non-simply connected space. So that for our illustration, we need another example of twisted product where the base is simply connected.

A popular example of this sort is the *Hopf fibration*: $S^1 \hookrightarrow S^3 \to S^2$. Think of S^3 as the unit sphere of $\mathbb{C}^2 = \mathbb{R}^4$, that is:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \ \underline{\mathbf{st}}\ |z_1|^2 + |z_2|^2 = 1\}. \tag{46}$$

The group $S^1 = \{z \in \mathbb{C}\ \underline{\mathbf{st}}\ |z| = 1\}$ acts over S^3 by the diagonal action $z \cdot (z_1, z_2) := (zz_1, zz_2)$, and the quotient of this action, in other words the orbit space, is homeomorphic to the sphere S^2: it is the classical result that the projective space $P^1\mathbb{C}$ is nothing but the Riemann sphere. Combined with the canonical inclusion $S^1 \hookrightarrow S^3 : z \mapsto (z, 0)$, we obtain the twisted product:

$$S^1 \hookrightarrow S^3 \to S^2 \tag{47}$$

which is the Hopf fibration. It can be proved this fibration actually can be organized as a twisted product as described Section 12, not so easy. Note this time the base space S^2 is not a group, for S^1 is not a normal subgroup of S^3; the 2-sphere S^2 is only a *homogeneous* space; but the sphere S^2 is simply connected and the so called Serre spectral sequence can be used.

This works as follows. First an array of groups $E^2_{p,q}$ is to be constructed, where every $E^2_{p,q}$ is made of two groups appearing in the Künneth theorem:

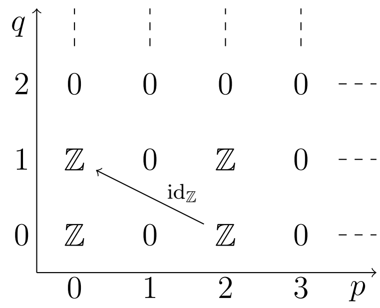
$$E^2_{p,q} = (H_p(B) \otimes H_q(F)) \oplus \ \mathrm{Tor}(H_{p-1}(X), H_q(F)). \tag{48}$$

In our case of the Hopf fibration, only four $E^2_{p,q}$ are non-trivial, namely $E^2_{0,0} = \mathbb{Z}$, $E^2_{0,1} = \mathbb{Z}$, $E^2_{2,0} = \mathbb{Z}$, $E^2_{2,1} = \mathbb{Z}$. The $E^2_{p,q}$'s can be organize as an array with two axes p and q, giving in our case:

$q \uparrow$	⋮	⋮	⋮	⋮	
2	0	0	0	0	---
1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	---
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	---
	0	1	2	3	$p \to$

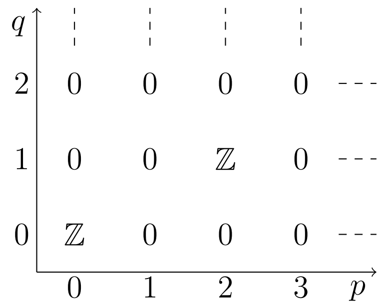
This diagram is traditionally called the E^2-page of the spectral sequence, the starting page for the Serre spectral sequence. If we add all the displayed groups along the diagonals $p+q = n$, we obtain the homology groups of the (non-twisted) product $S^1 \times S^2$, namely $H_n = \mathbb{Z}$ for $0 \leq n \leq 3$. But in the twisted case, the process is not finished, we must run all the next pages of the spectral sequence, starting from the displayed E^2-page. This consists in installing mysterious differentials $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$, mysterious, but of course well defined by the theory of spectral sequences, not detailed here[4]. In our case, please admit there is only one non-trivial differential, between $E^2_{2,0}$ and $E^2_{0,1}$, isomorphic to $\mathrm{id}_{\mathbb{Z}}$. No choice for the others $d^2_{p,q}$. We obtain the diagram:

[4] In this case, the non trivial $d^2_{2,0}$ is essentially the Chern class c_1 of the fibration.



E^2-page

These $d^2_{p,q}$'s really are differentials: the composition of two successive ones is null. This strange table of $d^2_{p,q}$'s is nothing but a collection of chain complexes, and the E^3-page is made of all the homology groups of these:



E^3-page

Again you must now install new differentials $d^3_{p,q} : E^3_{p,q} \to E^3_{p-3,q+2}$, but because of the geometry of our E^3-page, all these differentials are null, which implies the E^4-page is the same as the E^3-page. And so on up to the E^∞-page, the same as the E^3-page. There remains to sum along the diagonals $p + q = n$, giving $H_0(S^3) = H_3(S^3) = \mathbb{Z}$, the other homology groups being null and to compare with the formulas (31) to verify our computation is coherent with a previous one for the same groups.

Here, we were lucky, for in the general case, it is even not enough to sum the $E^\infty_{p,q}$ along the diagonals $p+q = n$, for these $E^\infty_{p,q}$'s are only the components of the graduated module associated to some filtration of the unknown group $H_n(F \times_\tau B)$, leading sometimes again to terrible extension problems.

7.5.4 Classifying spaces and loop spaces.

Did you think of this problem: Given a space X, is it possible to find an "inverse" space Y, that is, a space satisfying $X \times Y = 1$. The righthand term 1 should be the unit for the product, that is the point space $* = \Delta^0$ and the equation to be solved becomes $X \times Y = *$. Because of the definition of the product, if X is not also a point, it is clearly impossible. But we can ask the same question *up to homotopy*: is it possible to find a space Y such that the product $X \times Y \sim *$? If X is simply connected and non-contractible, it is impossible: certainly some homology groups $H_n(X)$ are non-null and the Künneth Theorem implies it is not possible to find a solution for Y, because it is clearly impossible to annihilate the non-null $H_n(X)$ when computing the $H_n(X \times Y)$. In the non-simply connected case, an analogous argument based over homotopy groups, in fact quite simpler, gives the same result. It seems there are no possible solutions for inversion in topology, except in trivial situations.

Yes, there are, but you must consider some *twisted* products.

Theorem 37 — *Let X be a connected space. Then there exists a* contractible *twisted product $\Omega X \times_\tau X$ with ΩX the loop space of X. Every space satisfying this condition has the homotopy type of ΩX.*

The product is contractible, so that up to homotopy the relation $\Omega X \times_\tau X \sim *$ is satisfied. This is true in the topological framework with an appropriate notion of fibration. It is true also in the simplicial framework and such a solution, due to Daniel Kan, is detailed in Section 9.2.

The loop space ΩX being defined in this way as a *left* inverse of the original space X, another natural question is now to design a method computing the homology groups $H_*\Omega X$ from the given groups H_*X. In fact the last groups are not enough, some further topological informations not contained in the homology groups are necessary, but the setting of these notes does not allow the author to go further along this problem. See [20, Section 9] for a detailed study of this problem, and a complete algorithmic solution.

Note also these twisted products are not symmetric: the respective roles of the fiber space and the base space are quite different, so that it is natural to ask also the symmetric question: given a space X, is it possible to find another space Y and a twisted product $X \times_\tau Y$? Examining this question leads to restrict the nature of the space X: it must be a topological group G or something analogous up to homotopy, and the solution Y is then called the classifying space BG.

Theorem 38 — *Let G be a topological group. Then there exists a* contractible *twisted product $G \times_\tau BG$ with a space BG called the classifying space of the group G. Every space satisfying this condition has the homotopy type of BG.*

A particular case is G the Eilenberg-MacLane space $G = K(\pi, n)$ for some commutative group π and some positive integer n; this Eilenberg-MacLane space is itself naturally provided with a structure of commutative group, in particular $\pi = K(\pi, 0)$, and a recursive definition of $K(\pi, n)$ is $K(\pi, n) = BK(\pi, n-1))$.

This space is called the classifying space, because it is essential when classifying the fibrations using the structural group G. In the discrete case, Section 5.2.1, a simplicial description of BG is given and also the twisting function τ. The case of $K(\pi, n)$ for π a discrete group is explained in Section 9.1.

Because G is a group, the chain complex C_*G is provided with a natural *product*, producing a structure of *differential algebra*. Eilenberg and MacLane proved this structure is the key point to determine the homology groups $H_*(BG)$.

7.5.5 Eilenberg-Moore spectral sequences.

The loop space ΩX of a connected space X and the classifying space BG of a topological group G are "inverse" spaces in the world of twisted products:

$$\Omega X \times_\tau X \sim * \qquad G \times_\tau BG \sim * \tag{49}$$

If a number x is close to the unit 1, an inverse x^{-1} of x can be computed as a sum of geometric series :

$$x^{-1} = 1 + (1-x) + (1-x)^2 + \cdots = \sum_{n=0}^{\infty}(1-x)^n \tag{50}$$

If X is simply connected and G connected, the homology groups of ΩX and BG can be computed essentially in the same way: the Eilenberg-Moore spectral sequences give sense to the formulas:

$$\begin{array}{l} H_*\Omega X = \sum_{n=0}^{\infty}(1 - H_*X)^n \\ H_*BG = \sum_{n=0}^{\infty}(1 - H_*G)^n \end{array} \tag{51}$$

In particular the expression $(1 - H_*X)$, for example, means you must cancel the $H_0X = \mathfrak{R}$ in H_*X, that is, only the "tail" $H_1t + H_2t^2 + \ldots$ must be considered in the total homology. The "negative" sign in this total homology is a consequence of a desuspension (loop space) or suspension (classifying space) process to be applied to the total homology. Furthermore, the infinite sum $\sum_{n=0}^{\infty}$ must be *twisted* too, using the Alexander-Whitney coproduct in C_*X, sketched in the next Section, or the product structure in C_*G coming from the group structure of G. Algorithmic versions of these spectral sequences are explained in [20], using the methods of *constructive* algebraic topology [18].

7.6 Alexander-Whitney coproduct and operads.

As explained in the previous section, a coalgebra structure, mainly defined by the Alexander-Whitney coproduct, can be installed on the chain complex associated to a simplicial set. The process is remarkably simple, so simple that it is a little strange this structure finally has so much importance, but this is the starting point of a long process leading to the modern and outstanding point of view of *operadic structure*. A good opportunity to present the *general* status of modern Algebraic Topology.

Definition 39 — Let X be a simplicial set. The Alexander-Whitney coproduct is a chain-complex morphism:

$$\Delta : C_*X \to C_*X \otimes C_*X \tag{52}$$

defined as follows for a generator $\sigma \in X_n$:

$$\Delta(\sigma) := \sum_{i=0}^{n} \partial_{i+1} \ldots \partial_n a \otimes \partial_0 \ldots \partial_{i-1} a. \tag{53}$$

Typically, if $X = \Delta^3$ and σ is the maximal simplex 0123 of Δ^3, then, with an obvious interpretation of the integer sequences:

$$\Delta(0123) = 0 \otimes 0123 + 01 \otimes 123 + 012 \otimes 23 + 0123 \otimes 3. \qquad (54)$$

It happens this is nothing but an *algebraic* version of the diagonal map $X \to X \times X : x \mapsto (x, x)$, interpretation valid thanks to the Eilenberg-Zilber equivalence $C_*(X \times X) \sim (C_*X \otimes C_*X)$. This coproduct is associative, of if you prefer co-associative, with an obvious definition, but *not commutative*. You could consider it is a drawback, but it is not. The induced map in cohomology is commutative, you could think it is an advantage, but it depends on the point of view.

It is a good opportunity to point out a terrible drawback of the very definition of the notion of simplicial set. Essentially, all the vertices of a simplex are *numbered*, from 0 to n for an n-simplex. This defines an order over the vertices, orientation of the edges, ortientation of triangles, and so on. But all these extra data are *arbitrary* and do not correspond at all with the pure notion of an "abstract" simplex where there is no reason to choose some order of the vertices: the very definition of a simplicial set leads to arbitrary choices in the description, which will terribly resctrict the scope of simple algebraic topology. You could object on the contrary the study by Eilenberg and Steenrod in [7, Chapter VI], already mentioned, allows you to close this matter? In a sense, simple algebraic topology makes commutative the non-commutative world of topology, but it is cheating, the non-commutativity of the topological world cannot be indefinitely hidden and this non-commutativity must finally be considered and processed.

It happens the first step in this direction is the Alexander-Whitney coproduct, which is *not* commutative, and in a sense which captures a (very small) part of the unavoidable non-commutativity of the topological world.

If you continue to study this matter about commutativity in Algebraic Topology, you will certainly go to the notion of *operadic structure* over a chain complex. The notion of E_∞ operad can be defined, an E_∞-structure makes sense for a chain complex and the final essential result along this line is the following.

Theorem 40 (Michael Mandell [16]) — *Let X be a simply connected homotopy type, all the $\mathbb{Z}$-homology groups of which have finite type. Let C_* be a free $\mathbb{Z}$-chain complex of finite type whose homology groups are those of X. Then it is possible to define an E_∞-structure over C_*, this structure being* equivalent *to the homotopy type.*

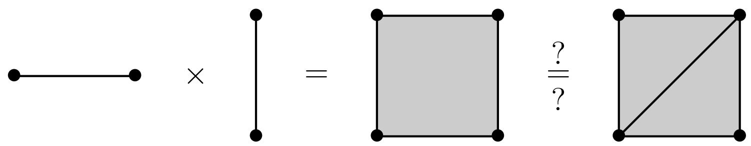
It was explained after Definition 28 the homology groups are not enough to identify a homotopy type, in fact are far from being enough. Mandell's theorem explains that completing the homology groups, more precisely a free chain complex having the right homology groups, with an E_∞-operadic structure allows you to provide the missing information to complete the definition of this homotopy type. In other words, any "reasonable" homotopy type can be defined by a chain complex of finite type provided with an E_∞-structure. The given homotopy type has been entirely "algebraized". It happens the inductive system of symmetric groups $\mathfrak{S} := (\mathfrak{S}_n)_{n\in\mathbb{N}}$ plays a major role in the definition of the E^∞-operad: the "non-commutative" component of a homotopy type is captured in an E_∞-structure.

But the next natural problem: *"Is it possible to* constructively *classify in this way the collection of the reasonable homotopy types?"* is today entirely *open*.

8 Products of simplicial sets.

The general work style in Algebraic Topology consists in firstly proving results for simple spaces, next deducing analogous results for more complicated spaces constructed from these spaces. The *product* constructor is important, as in most parts of mathematics, a more sophisticated one in topology being the *twisted product* constructor, invoking fibrations.

A simple but terrible observation is to be made about products, if one works in the simplicial framework: the product of two simplices *is not* a simplex. For example a 1-simplex is an interval, the product of two intervals is a square, which cannot be naturally identified to a 2-simplex. But this square can be divided into two triangles, that is, two 2-simplices, and we must carefully organize this remark, not so easy. We will see the simplicial set structure magically gives the right solution, rather amazing!



Definition 41 — If X and Y are two simplicial sets, the *simplicial product* $Z = X\times Y$ is defined by $Z_m = X_m \times Y_m$ for every natural number m, and $\alpha^*_Z = \alpha^*_X \times \alpha^*_Y$ if α is a Δ-morphism.

The definition of the product of two simplicial sets is perfectly trivial and is however at the origin of several landmark problems in algebraic topology, for example the deep structure of the twisted Eilenberg-Zilber theorem, still quite mysterious, and also the enormous field around the Steenrod algebras.

Every simplex of the product $Z = X \times Y$ is a *pair* (σ, τ) made of one simplex in X and one simplex in Y; both simplices must have the *same dimension*. It is tempting at this point, because of the "product" ambience, to denote by $\sigma \times \tau$ such a simplex in the product but *this would be a terrible error!* This is not at all the right point of view; the pair $(\sigma, \tau) \in Z_m$ is the unique m-simplex in Z whose respective *projections* in X and Y are σ and τ, *again* some m-simplices, and this is the reason why the pair notation (σ, τ) is the only one which is possible. For example the diagonal of a square is a 1-simplex, the unique 1-simplex the projections of which are both factors of the square; on the contrary, the "product" of the factors is simply the square, which does not have the dimension 1 and which is even not a simplex.

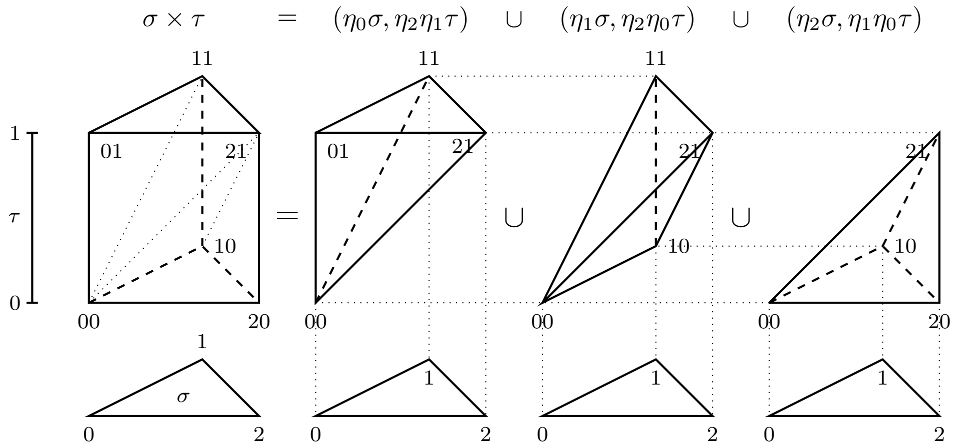
Theorem 42 — *If X and Y are two simplicial sets and $Z = X \times Y$ is their simplicial product, then there exists a canonical homeomorphism between $|Z|$ and $|X| \times |Y|$, the last product being the product of k-spaces.*

If you consider the product $|X| \times |Y|$ as the ordinary product of topological spaces, the same accident as for CW-complexes (see [8, p.59]) can happen. The framework of k-spaces avoids this obstacle, reducing the problem to finite simplicial subsets. Furthermore this esoteric problem does not exist when both factors are *countable* simplicial sets (countable sets of simplices), most often the case in concrete constructive topology.

♣ There are natural simplicial projections $X \times Y \to X$ and Y which define a canonical continuous map $\phi : |X \times Y| \to |X| \times |Y|$. The interesting question is to define its inverse $\psi : |X| \times |Y| \to |X \times Y|$.

First of all, let us detail the case of $X = \Delta^2$ and $Y = \Delta^1$ where the essential points are visible. The first factor X has dimension 2, and the second one Y has dimension 1 so that the product Z shoud have dimension 3. What about the 3-simplices of Z? There are 3 such *non-degenerate* 3-simplices, namely $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$, $\rho_1 = (\eta_1\sigma, \eta_2\eta_0\tau)$ and $\rho_2 = (\eta_2\sigma, \eta_1\eta_0\tau)$, if σ (resp. τ) is the unique non-degenerate 2-simplex (resp. 1-simplex) of Δ^2 (resp. Δ^1). This is nothing but the decomposition of a prism $\Delta^2 \times \Delta^1$ in three tetrahedrons.

Note no non-degenerate 3-simplex is present in X and Y and however some 3-simplices must be produced for Z; this is possible thanks to the *degenerate* simplices of X and Y where they are again playing a quite tricky role in our workspace; in particular a pair of *degenerate* simplices in the factors can produce a *non-degenerate* simplex in the product! This happens when there is no common degeneracy in the factors.



For example the tetrahedron $\rho_0 = (\eta_0\sigma, \eta_2\eta_1\tau)$ inside Z is *the* unique 3-simplex the first projection of which is $\eta_0\sigma$, and the second projection is $\eta_2\eta_1\tau$; the first projection is a tetrahedron collapsed on the triangle σ, identifying two points when the sum of barycentric coordinates of index 0 and 1 (the indices where injectivity fails in η_0) are equal; the second projection is a tetrahedron collapsed on an interval, identifying two points when the sum of barycentric coordinates of index 1, 2 and 3 are equal.

Let us take a point of coordinates $r = (r_0, r_1, r_2, r_3)$ in the simplex ρ_0. Its first projection is the point of $X = \Delta^2$ of barycentric coordinates $s = (s_0 = r_0 + r_1, s_1 = r_2, s_2 = r_3)$; in the same way its second projection is the point of $Y = \Delta^1$ of barycentric coordinates $t = (t_0 = r_0, t_1 = r_1 + r_2 + r_3)$. So that:

$$\phi(\rho_0, (r_0, r_1, r_2, r_3)) = ((\sigma, (r_0 + r_1, r_2, r_3)), (\tau, (r_0, r_1 + r_2 + r_3))) \tag{55}$$

In the same way:

$$\begin{array}{rcl} \phi(\rho_1, (r_0, r_1, r_2, r_3)) & = & ((\sigma, (r_0, r_1 + r_2, r_3)), (\tau, (r_0 + r_1, r_2 + r_3))) \\ \phi(\rho_2, (r_0, r_1, r_2, r_3)) & = & ((\sigma, (r_0, r_1, r_2 + r_3)), (\tau, (r_0 + r_1 + r_2, r_3))) \end{array} \tag{56}$$

The challenge then consists in deciding for an arbitrary point:

$$(\sigma, (s_0, s_1, s_2)), (\tau, (t_0, t_1)) \in |X| \times |Y|$$

what simplex ρ_i it comes from and what a good ϕ-preimage (ρ_i, r) could be. You obtain the solution in comparing the sums $u_0 = s_0$, $u_1 = s_0 + s_1$, $u_2 = t_0$; the sums $s_0 + s_1 + s_2$ and $t_0 + t_1$ are necesarily equal to 1 and do not play any role. You see in the three cases, the values of u_i's are:

$$\begin{array}{rl} ((\eta_0\sigma, \eta_2\eta_1\tau), r) \Rightarrow & u_0 = r_0 + r_1, u_1 = r_0 + r_1 + r_2, u_2 = r_0, \\ ((\eta_1\sigma, \eta_2\eta_0\tau), r) \Rightarrow & u_0 = r_0, u_1 = r_0 + r_1 + r_2, u_2 = r_0 + r_1, \\ ((\eta_2\sigma, \eta_1\eta_0\tau), r) \Rightarrow & u_0 = r_0, u_1 = r_0 + r_1, u_2 = r_0 + r_1 + r_2, \end{array} \tag{57}$$

so that you can guess the degeneracy operators to be applied to the factors σ and τ from the order of the u_i's; more precisely, sorting the u_i's puts the u_2 value in position 0, 1 or 2, and this gives the index for the degeneracy to be applied to σ; in the same way the u_0 and u_1 values must be installed in positions "1 and 2", or "0 and 2", or "0 and 1" and this gives the two indices (in reverse order) for the degeneracies to be applied to τ. It's a question of *shuffle*. Furthermore you can find the components r_i from the differences between successive u_i's. Now we can construct the map ψ:

$$\begin{array}{rcll} \phi((\sigma, s)(\tau, t)) & = & (\rho_0, (u_2, u_0 - u_2, u_1 - u_0, 1 - u_1)) & \text{if } u_2 \leq u_0 \leq u_1, \\ & = & (\rho_1, (u_0, u_2 - u_0, u_1 - u_2, 1 - u_1)) & \text{if } u_0 \leq u_2 \leq u_1, \\ & = & (\rho_2, (u_0, u_1 - u_0, u_2 - u_1, 1 - u_2)) & \text{if } u_0 \leq u_1 \leq u_2. \end{array} \tag{58}$$

There seems an ambiguity occurs when there is an equality between u_2 and u_0 or u_1, but it is easy to see both possible preimages are in fact the same in $|Z|$.

Now this can be extended to the general case, according to the following recipe. Let $\sigma \in X_m$ and $\tau \in Y_n$ be two simplices, $s \in \Delta_m$ and $t \in \Delta^n$ two geometric points. We must define $\psi((\sigma, s), (\tau, t)) \in |Z| = |X \times Y|$. We set $u_0 = s_0$, $u_1 = s_0 + s_1$, ..., $u_{m-1} = s_0 + \ldots + s_{m-1}$, $u_m = t_0$, $u_{m+1} = t_0 + t_1$, ..., $u_{m+n-1} = t_0 + \ldots + t_{n-1}$.

Then we sort the u_i's according to the increasing order to obtain a sorted list ($v_0 \leq \ldots \leq v_{m+n-1}$). In particular $u_m = v_{i_0}, \ldots, u_{m+n-1} = v_{i_{n-1}}$ with $i_0 < \ldots < i_{n-1}$, and $u_0 = v_{j_0}, \ldots, u_{m-1} = v_{j_{m-1}}$ with $j_0 < \ldots < j_{m-1}$. Then:

$$\begin{aligned} \psi((\sigma, s), (\tau, t)) &= ((\eta_{i_{n-1}} \ldots \eta_{i_0} \sigma, \eta_{j_{m-1}} \ldots \eta_{j_0} \tau), \\ &\quad (v_0, v_1 - v_0, \ldots, v_{m+n-1} - v_{m+n-2}, 1 - v_{m+n-1}))\ . \end{aligned} \tag{59}$$

Now it is easy to prove $\psi \circ \phi = \mathrm{id}_{|Z|}$ and $\phi \circ \psi = \mathrm{id}_{|X| \times |Y|}$, following the proof structure clearly visible in the case of $X = \Delta^2$ and $Y = \Delta^1$.

It is also necessary to prove the maps ϕ and ψ are continuous. But ϕ is the product of the realization of two simplicial maps and is therefore continuous. The map ψ is defined in a coherent way for each *cell* $\sigma \times \tau$ (this time it is really the *product* $|\sigma| \times |\tau| \subset |X| \times |Y|$) and is clearly continuous on each cell; because of the definition of the k-topology, the map ψ is continuous. ♣

If three simplicial sets X, Y and Z are given, there is only one natural map $|X \times Y \times Z| \to |X| \times |Y| \times |Z|$ so that "both" inverses you construct by applying twice the previous construction of ψ, the first one going through $|X \times Y| \times |Z|$, the second one through $|X| \times |Y \times Z|$ are necessarily the same: the ψ-construction is *associative*, which is interesting to prove directly; it is essentially the associativity of the Eilenberg-MacLane formula.

8.1 The case of simplicial groups.

Let G be a *simplicial group*. The object G is a simplicial object in the group category; in other words each simplex set G_m is provided with a group structure and the Δ-operators $\alpha^* : G_m \to G_n$ are group homomorphisms.

This gives in particular a continuous canonical map $|G \times G| \to |G|$; then identifying $|G \times G|$ and $|G| \times |G|$, we obtain a "continuous" group structure for $|G|$; the word *continuous* is put between quotes, because this does not work in general in the topological sense: this works always only in the category of k-spaces where the group structure is a map $|G| \times |G| \to |G|$, the source of which being evaluated in the k-space category; because of this definition of product, it is then true that $|G| \times |G| = |G \times G|$. The composition rule so defined on $|G|$ satisfies the group axioms; in particular the associativity property comes from the considerations about the associativity of the ψ-construction in the previous section.

9 Other examples of simplicial sets.

9.1 The Eilenberg-MacLane spaces.

The classifying spaces of groups, see Section 5.2.1, are particular cases of Eilenberg-MacLane spaces. Let G be some discrete group; the *realization* process applied to the simplicial set BG produces a model for $K(G,1)$: all the homotopy groups are null except π_1 canonically isomorphic to G. The construction can be generalized to construct $K(\pi,d)$, $d>1$, when π is an *abelian* group. See also [13, Chapter V] where these questions are carefuly detailed.

Let π be a fixed abelian group, and d a natural number. The simplicial set $E(\pi,d)$ is defined as follows. The set of m-simplices $E(\pi,d)_m$, shortly denoted by E_m, is $E_m = C^d(\Delta^m,\pi)$, the group of *normalized* d-cochains on the standard m-simplex with values in π. Such a cochain σ is nothing but a map $\sigma : \Delta^m_d \to \pi$, defined on the (degenerate or not) d-simplices of Δ^m, null for the degenerate simplices. If α is a Δ-morphism $\alpha : \underline{n} \to \underline{m}$, this map defines a simplicial map $\alpha_* : \Delta^n \to \Delta^m$ which in turns defines a pullback map $\alpha^* : C^d(\Delta^m,\pi) \to C^d(\Delta^n,\pi)$ between m-simplices and n-simplices of E_m.

The simplicial set $E(\pi,d)$ so defined contains the simplicial subset $K(\pi,d)$, made only of the *cocycles*, those cochains the coboundary of which ($d : C^d(\Delta^m,\pi) \to C^{d+1}(\Delta^m,\pi)$ is null. In fact $E(\pi,d)$ is a *simplicial group*, that is, a simplicial object in the category of groups, and $K(\pi,d)$ is a simplicial subgroup. The quotient simplicial group $E(\pi,d)/K(\pi,d)$ is canonically isomorphic to $K(\pi,d+1)$ and this structure defines the Eilenberg-MacLane fibration:

$$K(\pi,d) \hookrightarrow E(\pi,d) \to K(\pi,d+1) \tag{60}$$

See later the section about *simplicial fibrations* for some details.

9.2 Simplicial loop spaces.

Let X be a simplicial set. We can construct a new simplicial set $DT(X)$ (the acronym DT meaning Dold-Thom) from X, where $DT(X)_m$ is the free $\mathbb{Z}$-module generated by the m-simplices X_m; the operators of $DT(X)$ are also "generated" by the operators of X. This is a simplicial version of the Dold-Thom construction, producing a new simplicial set $DT(X)$, the homotopy groups of which being the homology groups of the initial X. The simplicial set $DT(X)$ is also a *simplicial group*; its simplex *sets* are nothing but the chain groups at the origin of the simplicial homology of X, but in $DT(X)$, each simplicial "chain" of X is *one* (abstract) simplex. See [13, Section 22].

The same construction can be undertaken, but instead of using the abelian group generated by the simplex sets X_m, we could consider the free *non-commutative* group generated by X_m. This also works, but then the obtained space is a simplicial model for the *James construction* of $\Omega\Sigma X$, the loop space of the (reduced) suspension of X. See [2] for the James construction in general and [4] for the simplicial case.

It is even possible to construct the "pure" loop space ΩX, without any suspension. This is due to Daniel Kan [12] and works as follows. It is necessary to assume X is reduced, that is with only one vertex: the cardinality of X_0 is 1. Let X^*_m the set of all m-simplices, except those that are 0-degenerate: $X^*_m = X_m - \eta_0(X_{m-1})$;

this makes sense for $m \geq 1$. Then let GX_m be the free *non-commutative* group generated by X^*_{m+1}; to avoid possible confusions, if $\sigma \in X^*_{m+1}$, let us denote by $\tau(\sigma)$ the corresponding *generator* of GX_m. The simplicial object GX to be defined is a simplicial *group*, so that it is sufficient to define face and degeneracy operators for the generators:

$$\begin{array}{rcll} \partial_i \tau(\sigma) & = & \tau(\partial_{i+1}\sigma), & \text{if } 1 \leq i \leq m; \\ \partial_0 \tau(\sigma) & = & \tau(\partial_1\sigma)\tau(\partial_0\sigma)^{-1}; & \\ \eta_i \tau(\sigma) & = & \tau(\eta_{i+1}\sigma), & \text{if } 0 \leq i \leq m. \end{array} \quad (61)$$

These definitions are coherent, and the simplicial set GX so obtained is a simplicial version of the loop space construction. See [13, Chapter VI] for details and related questions, mainly the *twisted Eilenberg-Zilber theorem*, at the origin of the general solution described in [20] for the computability problem in algebraic topology.

9.3 The singular simplicial set.

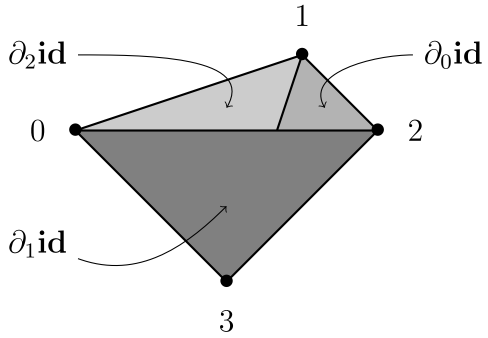
Let X be an arbitrary topological space. Then the *singular simplicial set* associated to X is constructed as follows. The set of m-simplices SX_m is made of the continuous maps $\sigma : \Delta^m \to X$; *one* (abstract) simplex is *one* continuous map but no topology is installed on SX_m; in particular when SX will be *realized* in the following section, the *discrete* topology must be used. The source of the abstract m-simplex σ is the geometric m-simplex $\Delta^m \subset \mathbb{R}^{\underline{m}}$ provided with the traditional topology. If $\alpha \in \Delta(\underline{n}, \underline{m})$ is a Δ-morphism, this α defines a natural continuous map $\alpha_* : \Delta^n \to \Delta^m$ between geometric simplices, and this allows us to naturally define $\alpha^*(\sigma) = \sigma \circ \alpha_*$. An enormous simplicial set is so defined if X is an arbitrary topological space; it is at the origin of the *singular homology* theory.

10 Kan extension condition.

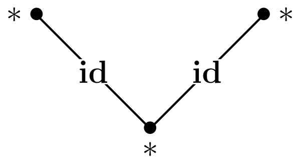
Let us consider the standard simplicial model S^1 of the circle, with one vertex and one non-degenerate 1-simplex σ. This unique 1-simplex clearly represents a generator of $\pi_1(S^1)$, but its double cannot be so represented. This has many disadvantages and correcting this defect was elegantly solved by Kan.

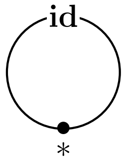
Definition 43 — A *Kan (m,i)-hat* (Kan hat in short) in a simplicial set X is a $(m+1)$-tuple $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ satisfying the relations $\partial_j \sigma_k = \partial_{k-1}\sigma_j$ if $j < k$, $j \neq i \neq k$.

For example the triple $(\partial_0\mathbf{id}, \partial_1\mathbf{id}, \partial_2\mathbf{id},)$ is a Kan $(2,3)$-hat in the standard 3-simplex Δ^3 if $\mathbf{id}$ is the unique non-degenerate 3-simplex, see Section 4.2. In the figure below representing this hat, the faces 013, 023 and 123 are all the 2-dimensional faces except one (012) of the 3-simplex $\mathbf{id}$.



In general, a *Kan* $(m, i)-hat$ is organized as all the faces except one, the i-th one, of a *potentially existing* simplex of dimension $m+1$, but this hypothetic simplex maybe in fact does not exist. In the simplex example above, the "hypothetic" simplex does exist, it is **id**. But let us now consider the following example for the circle S^1, this circle being modelled as a simplicial set as described in Section 4.3.





The righthand figure is the circle drawn as usual, showing the base point '$*$' and the fundamental 1-simplex **id**. The lefthand figure represents the 1-hat $(\mathbf{id}, \mathbf{id})$, but this definition is *incomplete*, an important point to be detailed now.

The first example $(\partial_0\mathbf{id}, \partial_1\mathbf{id}, \partial_2\mathbf{id})$ in Δ^3 is a correct Kan (2,3)-hat, but it is *not* a Kan (2,2)-hat. Why? In this case the definition of this (2,2)-hat would be:

$$\begin{aligned} \sigma_0 &= \partial_0\mathbf{id} \\ \sigma_1 &= \partial_1\mathbf{id} \\ \sigma_{\underline{3}} &= \partial_{\underline{2}}\mathbf{id} \end{aligned} \tag{62}$$

where the critical indices are underlined. Because we are considering whether our triple is a (2-2)-hat, the coherence condition of Definition 43 must in particular be satisfied for $j = 1$ and $k = 3$: $\partial_1\sigma_{\underline{3}} \stackrel{?}{=} \partial_2\sigma_1$, which becomes $\partial_1\partial_{\underline{2}}\mathbf{id} \stackrel{?}{=} \partial_2\partial_1\mathbf{id}$ but the righthand member of this relation is $\partial_1\partial_3\mathbf{id}$ and this relation is false: it would contradict Proposition 7.

Let us consider now the possible hat $(\mathbf{id}, \mathbf{id})$ of the circle S^1; every face of the fundamental simplex **id** is the base point '$*$', so that the coherence conditions of Definition 43 are certainly satisfied: our pair $(\mathbf{id}, \mathbf{id})$ is *as you like* a $(1,0)$-hat, a $(1,1)$-hat and a $(1,2)$-hat as well.

Definition 44 — If $(\sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1})$ is a Kan (m, i)-hat in the simplicial set X, a *filling* of this hat is a simplex $\sigma \in X_{m+1}$ such that $\partial_j\tau = \sigma_j$ for $j \neq i$.

The "fundamental" 3-simplex **id** of Δ^3 is a filling of the example Kan $(2,3)$-hat: $(\partial_0\mathbf{id}, \partial_1\mathbf{id}, \partial_2\mathbf{id})$ in Δ^3.

Let us now consider the hat $(\mathbf{id}, \mathbf{id})$ of S^1. Then $\eta_1\mathbf{id}$ fills the $(1,\underline{0})$-hat $(\mathbf{id}, \mathbf{id})$, for $\partial_1\eta_1\mathbf{id} = \mathbf{id}$ and $\partial_2\eta_1\mathbf{id} = \mathbf{id}$ as well. In the same way, $\eta_0\mathbf{id}$ fills the $(1,\underline{2})$-hat $(\mathbf{id}, \mathbf{id})$, for $\partial_0\eta_0\mathbf{id} = \partial_1\eta_0\mathbf{id} = \mathbf{id}$. But on the contrary the $(1,\underline{1})$-hat $(\mathbf{id}, \mathbf{id})$ *cannot* be filled. Only three 2-simplices in S^1, all degenerate, namely $\eta_1\eta_0*$, $\eta_0\mathbf{id}$ and $\eta_1\mathbf{id}$. But:

$$\begin{aligned} \partial_0\eta_1\eta_0\mathbf{id} &= \eta_0* \neq \mathbf{id} \\ \partial_2\eta_0\mathbf{id} &= \eta_0* \neq \mathbf{id} \\ \partial_0\eta_1\mathbf{id} &= \eta_0* \neq \mathbf{id} \end{aligned} \qquad (63)$$

and no 2-simplex is able to fill our $(1,1)$-hat. You see the second index in the definition of a hat is crucial.

Definition 45 — A simplicial set X satisfies the *Kan extension condition* if any Kan hat has a filling.

The standard simplex Δ^d satisfies the Kan condition. The other elementary simplicial sets in general do not, for example the spheres as described in Section 4.3.

The simplicial sets satisfying the Kan extension condition have numerous interesting properties; for example their homotopy groups can be combinatorially defined, see the next section, a canonical *minimal* version is included, also satisfying the extension condition [13, Section 9], a simple decomposition process produces a Postnikov tower [13, Section 8].

The simplicial groups are important from this point of view: in fact a simplicial group always satisfies the Kan extension condition [13, Theorem 17.1]. For example the simplicial description of $P^\infty\mathbb{R}$ (see Section 5.2) is a simplicial group and therefore satisfies the Kan condition, which is not so obvious; it is even minimal. The singular complex SX of a topological space X also satisfies the Kan condition but in general is not minimal. These simplicial sets satisfying the Kan condition are so interesting that it is often useful to know how to *complete* an arbitrary given simplicial set X and produce a new simplicial set X' with the same homotopy type satisfying the Kan condition. The Kan-completed X' can be constructed as follows.

Let us define first an elementary completion $\chi(X)$ for X. For each Kan (m,i)-hat of X, we decide to add the hypothetical $(m+1)$-simplex (even if a "solution" preexists), and the "missing" i-th face; such a completion operation does not change the homotopy type of X. Doing this completion construction for every Kan hat of X, we obtain the first completion $\chi(X)$. Then we can define $X_0 = X$, $X_{i+1} = \chi(X_i)$ and $X' = \lim_{\rightarrow} X_i$ is the desired Kan completion. You can also run this process in considering only the failing hats.

11 Homotopy groups.

The simplest invariants allowing one to distinguish different homotopy types are homology groups and homotopy groups. The initial definition of homotopy groups is due to Hurewicz, and is strangely much simpler than the definition of homology groups, due many years before to Poincaré. Another strange property is that, except in special cases, these homotopy groups are hard to be computed.

11.1 The topological case.

11.1.1 The Poincaré group.

The definition of Hurewicz goes as follows. The case of the first homotopy group $\pi_1 X$ of a pointed topological space $X = (X, *)$ is a little simpler and we start with this particular case. This group was already considered by Poincaré and is often called the Poincaré group of X. A pointed topological space is a space where some point $* \in X$ is distinguished. This point will be used as the mandatory starting and arriving points of the *loops*.

The pointed circle $S^1 = (S^1, *)$ is the usual unit circle of the complex plane $\mathbb{C}$ pointed at the unit $* = 1 \in \mathbb{C}$. A (topological) loop $\Omega X \ni \gamma : (S^1, *) \to (X, *)$ is a continuous map sending the base point of S^1 to the base point of X. Installing the compact-open topology over ΩX defines a structure of topological space; don't be anxious with the technicalities of such a topology, we will soon switch to a combinatorial framework where this esoteric subject disappears. It is natural to decide the trivial constant loop $* : S^1 \to X : *(t) :\equiv *$ is the base point of ΩX, so that the process can be iterated, defining $\Omega^n X$ for arbitrary $n \in \mathbb{N}$.

Two loops γ_0 and γ_1 are homotopic if a continuous map $h : I \times S^1 \to X$ can be installed between them; on one hand $h_{|\{i\} \times S^1} = \gamma_i$ for $i = 0, 1$; but the "intermediary" loops must also be… loops, that is, $h_{|I \times \{*\}} \equiv * \in X$. The homotopy relation '$\sim$' is an equivalence relation, allowing us to define the quotient $\pi_1 X := \Omega X / \sim$.

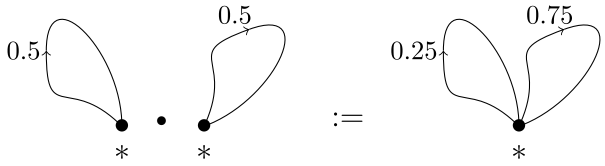
Poincaré discovered a natural group structure can be installed over $\pi_1 X$. If $\gamma, \delta \in \Omega X$, the composition $\varepsilon = \gamma \cdot \delta$ is defined by:

$$\begin{aligned} \varepsilon(z) &= \gamma(z^2) \text{ if } \Im z \geq 0, \\ &= \delta(z^2) \text{ if } \Im z \leq 0. \end{aligned} \tag{64}$$

You may not like this parametrization of the circle by the complex numbers of modulus 1. If you prefer a parametrization by the interval $I = [0, 1]$, often considered as a time interval, deciding that $z = e^{2\pi i t}$, you obtain:

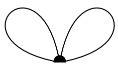
$$\begin{aligned} \varepsilon(t) &= \gamma(2t) && \text{if } t \in [0, 0.5], \\ &= \delta(2t-1) && \text{if } t \in [0.5, 1]. \end{aligned} \tag{65}$$

The following figure illustrates this composition.



where the "time" parametrization of the circle is used, and the surrounding space X is not shown.

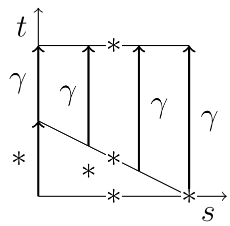
For later generalization, a dual point of view is convenient. The wedge $S^1 \vee S^1$ of two circles is the (categorical) sum of two circles in the category of the pointed spaces; it is the *disjoint* union of two circles with the base points identified:

$S^1 \vee S^1 :=$ 

Two canonical loops λ_1 and λ_2 of this wedge are the lefthand and righthand inclusions $S^1 \hookrightarrow S^1 \vee S^1$. The *composition* of these loops is a loop $\lambda = \lambda_1 \cdot \lambda_2 : S^1 \to S^1 \vee S^1$ running the whole wedge $S^1 \vee S^1$. Also, two loops $\gamma, \delta \in \Omega X$ define a continuous map $\gamma \vee \delta : S^1 \vee S^1 \to X$, this is the universal property of the categorical sum. And finally $\gamma \cdot \delta := (\gamma \vee \delta) \circ \lambda$. In this way the unique composition of λ_1 and λ_2 is extended to compositions of arbitrary loops.

The composition of loops *is not* associative, but is associative *up to homotopy*. It is a matter of *continuous* change of parametrization, from the time slicing $[0.00, 0.25, 0.50, 1.00]$ to $[0.00, 0.50, 0.75, 1.00]$.

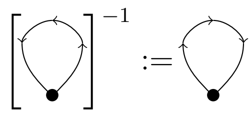
The unit of $\pi_1 X$ is the homotopy class '$*$' of the constant loop '$*$'. It is *not* a unit in ΩX for the composition, for in general, except if $\gamma = *$ is the constant loop, $* \cdot \gamma \neq \gamma$; but the loop $* \cdot \gamma$ is *homotopic* to γ. The proof is the following figure:



which produces the formulas:

$$\begin{aligned} h(s,t) &:= \gamma(\frac{s+2t-1}{s+1}) \quad \text{if } (1-s)/2 \leq t \leq 1; \\ &:= * \quad \text{otherwise.} \end{aligned} \tag{66}$$

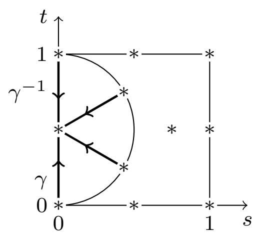
A non-constant loop has no inverse, but has a canonical inverse up to homotopy, consisting in running the same path, but in the opposite direction: $\gamma^{-1}(t) := \gamma(1-t)$.



The composition $\gamma \cdot \gamma^{-1}$ is homotopic to the constant loop, thanks to the map $h : I \times S^1 \to X$ defined by the formulas:

$$\begin{array}{rcll} h(s,t) & := & \gamma(1 - 2\sqrt{(1/2-t)^2 + s^2})) & \text{if } (1/2-t)^2 + s^2 \leq 1/4, \\ & := & * & \text{otherwise.} \end{array} \tag{67}$$

The following figure explains these formulas:



In this way, the restriction to the lefthand edge of the square, in fact a circle S^1, is the composition $\gamma \circ \gamma^{-1}$, while the restriction to the righthand edge is the constant loop at the base point.

The quotient $\pi_1 X := \Omega X/\sim$ inherits from loop composition a group structure, this is the Poincaré group of X, more precisely of the *pointed* space $(X, *)$.

11.1.2 Higher homotopy groups.

The construction of $\pi_1 X$ sketched in the previous section can be extended to higher dimensions, producing a collection of groups $\pi_n X$ for every integer $n \geq 0$. In the very particuliar case $n = 0$, it is natural to decide $\pi_0 X$ is the *set* of the path connected components of X, pointed by the component containing the base point. But except in particular situations, there is no natural group structure over this set $\pi_0 X$.

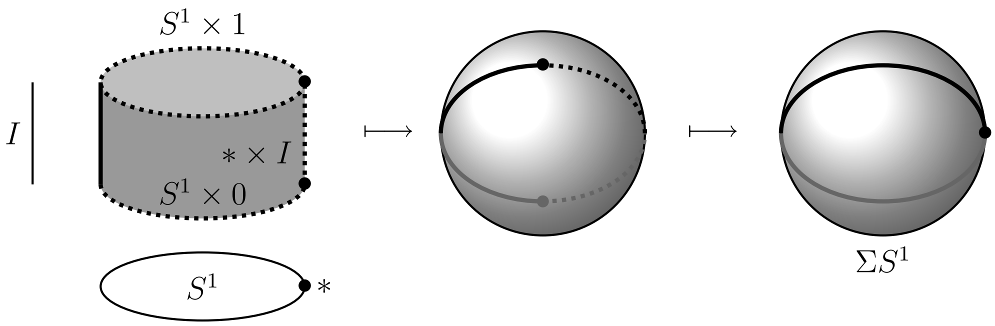
The most direct definition of $\pi_n X$ for $n \geq 2$ is simply $\pi_n X := \pi_1(\Omega^{n-1} X)$. We give now an equivalent definition closer to the particular definition of the π_1 group.

In the previous section, the composition of loops has in particular been defined through a particular map $\lambda : S^1 \to S^1 \vee S^1$.

Definition 46 — Let $(X, *)$ be a pointed space. Then the (pointed) *suspension* of X is the space:

$$\Sigma X := \frac{X \times I}{(X \times 0) \cup (X \times 1) \cup (* \times I)}. \tag{68}$$

For example the suspension of the pointed circle $(S^1, *)$ is the pointed 2-sphere $(S^2, *)$.

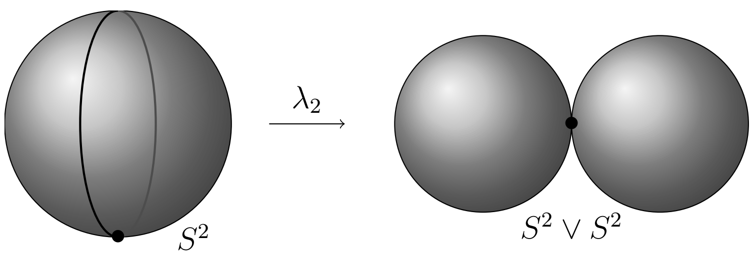


If you collapse over one point only the upper and lower circles $S^1 \times 1$ and $S^1 \times 0$, you already obtain a 2-sphere where the interval $* \times I$ maybe is become the Greenwich meridian. There remains to collapse also this meridian over a unique point, a sort of unique pole of the suspension ΣS^1, the base point of this suspension. The symmetric meridian which crossed the Pacific ocean is become a "meridian" running a whole big circle of the final sphere.

In general, any n-sphere is the suspension of the previous sphere: $(S^n, *) = \Sigma(S^{n-1}, *)$. More generally, the suspension operator Σ is a *functor*: if $f : (X, *) \to (Y, *)$ is a *pointed* map, that is $f(*) = *$, then a natural process constructs $\Sigma f : \Sigma(X, *) \to \Sigma(Y, *)$.

The suspenction functor commutes with the wedge operator: there is a canonical homeomorphism $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$. Exercise.

So that if we apply the suspension functor to the map $\lambda : S^1 \to S^1 \vee S^1$, the loop which runs both components of $S^1 \vee S^1$, then a new map is produced $\lambda_2 : S^2 \to S^2 \vee S^2$. Both meridians Greenwich and its symmetric are collapsed over the "central" base point of $S^2 \vee S^2$.



In the same way, canonical maps $\lambda_n : S^n \to S^n \vee S^n$ are defined for every integer $n > 0$.

It is natural to define $\pi_n X$ as the set of homotopy classes of pointed maps: $\pi_n X := \mathcal{C}((S^n, *), (X, *))/ \sim$. A multiplicative structure is defined as in dimension 1: $\gamma \cdot \delta = (\gamma \vee \delta) \circ \lambda_n$. It is not hard to proof a group structure is so defined; the group so obtained is in fact canonically isomorphic to $\pi_1 \Omega^{n-1} X$.

Proposition 47 — *Let X be a pointed topological space. Then the homotopy groups $\pi_n X$ are abelian for $n \geq 2$.*

On the contrary, the group π_1 in general is not commutative. For example, $\pi_1(S^1 \vee S^1)$ is the free non-commutative group with two generators.

Proposition 47 is a particular case of a more general result.

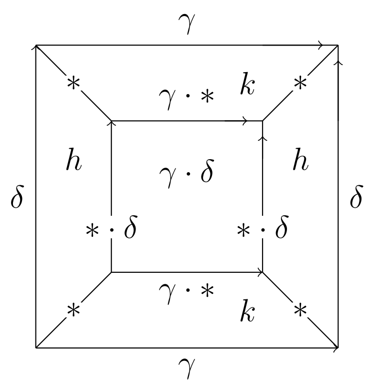
Proposition 48 — *Let $(X, *)$ be a connected space provided with a product $(x, y) \mapsto x \cdot y$ satisfying the following properties:*

- $* \cdot * = *$;
- *A homotopy map $h : I \times X \to X$ satisfies $h(0, x) = x$, $h(1, x) = * \cdot x$ and $h(s, *) \equiv *$: the left-product by $*$ is homotopic to the identity.*
- *A homotopy map $k : I \times X \to X$ satisfies $h(0, x) = x$, $h(1, x) = x \cdot *$ and $k(s, *) \equiv *$: the same for the right-product.*

Then the group $\pi_1 X$ is commutative.

In particular a *topological group* certainly satisfies these hypotheses.

♣ The proof is the figure:



which defines a map $I \times I \to X$. The restriction to the twelve segments of the figure is clearly shown. How to fill in the central square and the four trapezoids is also clearly indicated. This figure illustrates why the path $\gamma \cdot \delta =$ ⌟↑ is homotopic to the path $\delta \cdot \gamma =$ ↑⌝. ♣

♣ [Proposition 47] If $n \geq 1$, $\Omega^n X := \Omega(\Omega^{n-1} X)$ is provided with a product satisfying the properties required in Proposition 48: it is the path composition. Therefore $\pi_1 \Omega^n X =: \pi_{n+1} X$ is a commutative group for $n \geq 1$. ♣

11.2 Computability obstacles.

In general, even the computation of a $\pi_1 X$ is not so easy. The two following results, stated without any proof, are in this respect rather striking.

Theorem 49 — *There does not exist a general decision algorithm:*

- Input: *A finite simplicial complex K.*

- Output: *A boolean* $\beta \in \{\bot, \top\}$*, with* $\beta = \top$ *if and only if* $\pi_1 K = 0$.

If X is connected and its Poincaré group $\pi_1 X$ is null, the space X is said *simply connected*. It happens no general algorithm can decide whether this property is true or not, even for "simple" spaces such as the realizations of *finite* simplicial complexes. More precisely there exists a "*semi*-algorithm" α: if X is simply connected, then the algorithm α working over X will terminate and prove $\pi_1 X = 0$. If $\pi_1 X \neq 0$, then the algorithm does not terminate, but the user of this algorithm will not be "informed"!

The set of simplicial complexes with k given vertices is finite and the isomorphism problem between them is easily solved. Considering successively the isomorphism classes of simplicial complexes with 0 vertex, 1 vertex, 2 vertices, ..., k vertices, and so on, produces a countable list without repetition of all the isomorphism classes of the finite simplicial complexes.

Theorem 50 — *The set of isomorphism classes of simplicial complexes is countable, and can be presented as a sequence* $(K_n)_{n\in\mathbb{N}}$*. Then there exists an integer* $n_0 \in \mathbb{N}$ *satisfying:*

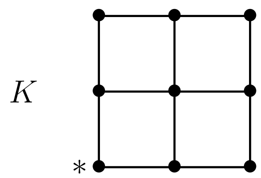
- *For every* $n < n_0$*, there exists a demonstration proving or disproving the relation* $\pi_1 K_n = 0$*;*
- $\pi_1 K_{n_0} \neq 0$*, but there des not exist a proof of this fact.*

These strange results are consequences of the Novikov theorem about the undecidability of the word problem in Group Theory. Note the sequence $(K_n)_{n\in\mathbb{N}}$ can be explicitly constructed, for example by a computer program, but this integer n_0 will remain *definitively* "hidden". When the mathematicians will try to study $\pi_1 K_{n_0}$, they will not succeed in proving $\pi_1 K_{n_0} = 0$, for it is false (!), but they will not succeed either in proving $\pi_1 K_{n_0} \neq 0$, because such a proof does not exist! And our mathematicians will definitively remain *in doubt*, unable to decide if this n_0 is this terrible number, or if more ordinarily they have not yet been skillful enough to elucidate this mystery. This a striking avatar of the famous Gödel's incompleteness theorem, which has quite concrete consequences!

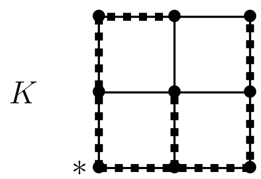
11.3 A recipe for "computing" the Poincaré group of a finite simplicial complex.

This title seems to contradict the theorems stated in the previous section! On the contrary this section will settle more clearly the obstacle, which illustrates also the frequent confusions about the notion of "result" in Mathematics.

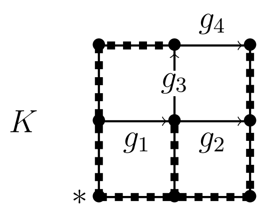
Let us consider the simple simplicial complex K made of 12 edges organized as four contiguous squares.



No triangle in this simplicial complex, the squares are hollow. The base point is the down left vertex. The first step to "compute" the Poincaré group consists in drawing a *maximal tree* rooted at the base point: this consists in taking the maximum number of edges, connected to the base point by some path, but *without introducing any cycle*. Such a maximal tree, shown by thick dashes overloading the corresponding edges on this figure, could be for K:



You observe adding any edge to this tree would generate a cycle, which is forbidden in a *tree*. There remains to denote the *other* edges, those which are complementery to this tree and, important, to orient them.



Then the group $\pi_1(K,*)$ is canonically isomorphic to the free non-commutative group generated by these particular edges: $\pi_1(K,*) =< g_1, g_2, g_3, g_4| \quad >$. The recipe to obtain a representant of these generators is simple: start from the base point, go to the *origin* of the marked edge *along the maximal tree*, a unique path, run the edge, and go back to the base point again along the chosen tree. For example the loop represented by the generator g_4 runs the perimeter of the whole square, in the clockwise sense.

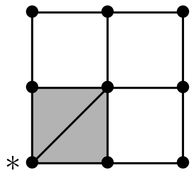
A non-commutative free group G of finite type admits many bases, but with the same cardinality n, for every basis of n elements induces an isomorphism $G/[G,G] \cong \mathbb{Z}^n$ and $\mathbb{Z}^m \cong \mathbb{Z}^n$ implies $m = n$. We obtain in this way a proof that the number of edges of a maximal tree is independent of the choice of this tree. The same if you change the base point in the same connected component:

Proposition 51 — *Let X be a connected topological space and $x_0, x_1 \in X$. Then every path connecting x_0 and x_1 in X induces a canonical isomorphism $\pi_1(X,x_0) \cong \pi_1(X,x_1)$.*

♣ Exercise. ♣

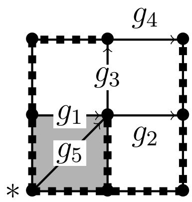
Our recipe is valid only for a 1-dimensional simplicial complex, without any triangle. Let us decide to fill the down-left square by two triangles to obtain the new simplicial complex K':

K'



Because of the triangles, one further edge is needed. The maximal tree can be the same as before, which could give the figure:

K'



We have now five generators, because of the added edge. But each triangle is the source of one *relation* in the resulting group, consisting in expressing the path running the perimeter of a triangle is homotopic to the constant loop, that is, the "null" loop. If we run the triangle perimeters in the counterclockwise sense, we obtain here the two relations: $g_5^{-1} = *$ and $g_5 g_1^{-1} = *$. In the traditional notation of finitely presented groups, we obtain:

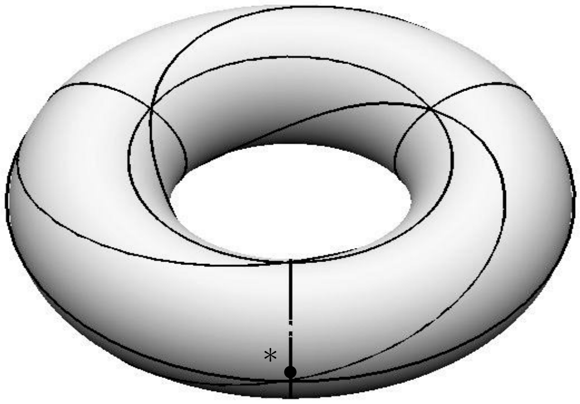
$$\pi_1(K', *) = < g_1, \ldots, g_5 | g_5^{-1}, g_5 g_1^{-1} > \tag{69}$$

But the first relation simply cancels the generator g_5 and the second one in turn cancels the generator g_1, so that finally:

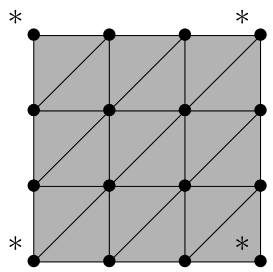
$$\pi_1(K', *) = < g_2, g_3, g_4 | > \tag{70}$$

The role of the filled square is simply, with respect to the initial situation with K, to cancel the generator g_1, for the loop running the boundary of the added square is now homotopic to the trivial loop.

These explanations should be enough for the general case. You do not have to take account of the possible simplices in dimensions ≥ 3. To be sure you have understood, consider the natural triangulation of the 2-torus $T_2 = S^1 \times S^1$ by 18 triangles:

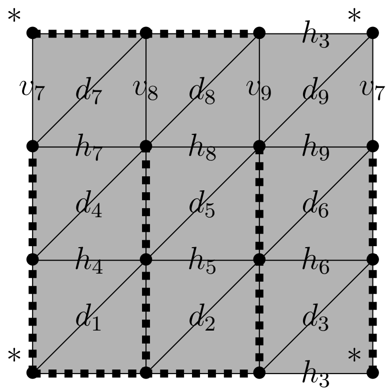


maybe more easily described by the planar figure:



Pay attention to the *repetitions* in this figure, where on one hand the left and right edges of the total square are to be identified, which gives maybe the torus meridian in front of us on the 3D figure; in the same way the upper and lower edges are to be identified, giving some "parallel" of our torus; so that finally the *four* corners of this representation represent in fact a *unique* point of the torus, maybe the basepoint highlighted on the 3D figure.

The preparatory work to compute $\pi_1(T_2)$ could be this figure:



The maximal tree is made in fact of only 8 edges; adding any other edge would generate a cycle, in particular for example if you finish the lefthand edge of the global square, because the down left and up left corners represent the same point of the torus, the base point: adding this edge would produce a *whole* torus meridian, certainly a cycle.

27 edges in our simplicial complex, so that the 19 remaining edges are the generators of our group. With obvious conventions in naming the edges and orienting them, we find:

$$\pi_1(T_2) = < h_3, \ldots, h_9, d_1, \ldots, d_9, v_7, v_8, v_9 | R > \tag{71}$$

where the relations R are made of 18 elements, one for each triangle:

$$\begin{array}{cccccc} d_1^{-1}, & d_2^{-1}, & h_3 d_3^{-1}, & d_1 h_4^{-1}, & d_2 h_5^{-1}, & d_3 h_6^{-1}, \\ h_4 d_4^{-1}, & h_5 d_5^{-1}, & h_6 d_6^{-1}, & d_4 h_7^{-1}, & d_5 h_8^{-1}, & d_6 h_9^{-1}, \\ h_7 v_8 d_7^{-1}, & h_8 v_9 d_8^{-1}, & h_9 v_7 d_9^{-1}, & d_7 v_7^{-1}, & d_8 v_8^{-1}, & d_9 h_3^{-1} v_9^{-1}. \end{array} \tag{72}$$

But we can use these relations to cancel successively many generators, for example:

$$d_1, d_2, h_4, h_5, d_4, d_5, h_7, h_8. \tag{73}$$

Also:

$$v_7 = d_7 = v_8 = d_8 = v_9 \text{ and } h_3 = d_3 = h_6 = d_6 = h_9 \tag{74}$$

which leads to the simplified presentation:

$$< h_3, v_7, d_9 | d_9 = v_9 h_3 = h_3 v_9 > = < h, v | hv = vh > \tag{75}$$

It is the group with two *commuting* generators. In other words $\pi_1 T_2 = \mathbb{Z}^2$. Note that S^1 is the *group* of the complex numbers of modulus 1, and $T_2 = S^1 \times S^1$ can be provided with the product group structure. In particular T_2 satisfies the properties of Proposition 48, and the commutativity of the π_1 was in this case *mandatory*.

How make compatible the relatively simple method described in this section and the negative results of the previous section? The process described here allows one to easily give a *finite presentation* of the Poincaré group of any finite simplicial complex, but a problem remains: such a finite presentation is in a sense a *partial* result. For example the isomorphism problem between such groups is in general undecidable: no general algorithm can consider two finite presentations Pr_1 and Pr_2 respectively "describing" some groups G_1 and G_2 and then decide whether these groups are isomorphic or not. And claiming that our combinatorial method "computes" the Poincaré group has in fact a limited scope.

The problem would be solved if, given some finite presentation Pr of some group, it would exist a canonical process to obtain from this maybe complicated presentation the *simplest one*, a process usually called the search of a *normal form* among equivalent presentations. It fortunately or unfornately happens, depending of the point of view, such a normalization process in general does not exist, otherwise the Novikov theorem about the undecidability of the word problem would be false.

11.4 Analogous methods in higher dimensions?

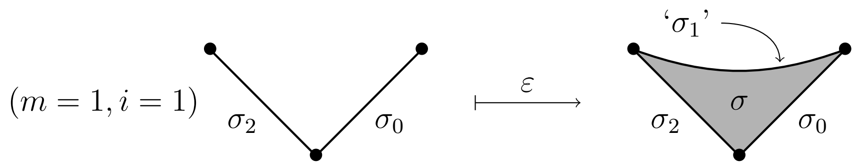
Methods of this sort do not exist! The combinatorial methods describing the higher homotopy groups exist only for the Kan simplicial sets, and these simplicial sets, even for a very simple initial simplicial set, are not in general of finite type. For example any Kan simplicial set having the homotopy type of the 2-sphere S^2 has an infinite number of non-degenerate simplices of dimension n for any dimension $n \geq 2$. This is essentially caused by the deep *topological* nature of these homotopy groups. These difficulties are from another respect paradoxical : these higher homotopy groups are necessarily commutative, and of finite type for any "reasonable" space, a striking result due to Jean-Pierre Serre, thanks to his famous spectral sequence.

We nevertheless describe here the main components of the combinatorial definition of the homotopy groups, due to Daniel Kan. This definition cannot be directly used for computations, but is useful in many cases, typically to easily understand the notion of Postnikov tower.

We work with a pointed Kan simplicial set $(K, *)$, the base point $*$ being some vertex $* \in K_0$, the set of 0-simplices. Definition 45 ensures the existence of some extension function ε:

$$\varepsilon : (m, i, \sigma_0, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{m+1}) \mapsto \sigma \tag{76}$$

where the input is a Kan (m, i)-hat (see Definition 43) and the outpout is a filling of this hat (Definition 44). The component m is redundant in the input but the component i is not! Look at the discussion p.51 to understand why this component is important.



This extension function is a sort of "simplex composition": when $m+1$ m-simplices are coherently contiguous to each other, then the filling produces another m-simplex to be considered as the composition of the input simplices. In the simplest case of the figure above, the edge σ_1 is the composition of σ_0 and σ_2 produced by ε; the justification of this composition is the simplex σ to be considered as a *homotopy* between $\sigma_0 \cdot \sigma_2$ and σ_1.

Definition 52 — An *n-sphere* of the pointed simplicial complex $(K, *)$ is an n-simplex $\sigma \in K_n$ satisfying $\partial_i \sigma = *_{n-1}$ for $0 \leq i \leq n$.

More precisely $\partial_i \sigma = \eta_{n-2} \cdots \eta_0 * =: *_{n-1}$: every face of our n-sphere is the base point degenerated up to dimension $n-1$. Such a sphere defines a canonical simplicial map $\gamma : S^n \to K$ from the n-sphere, presented as a simplicial set as explained Section 4.3, toward our simplicial set K: the fundamental simplex of S^n is sent over σ and this is enough to define our simplicial map.

Proposition 53 — *Let σ and σ' be two n-spheres of the Kan pointed simplicial set $(K, *)$. Then $(n, n, *_n, \ldots, *_n, \sigma, \sigma')$ is a Kan (n, n)-hat. This allows to define* *the composition of these spheres:*

$$\sigma \cdot \sigma' := \partial_n \varepsilon(n, n, *_n, \ldots, *_n, \sigma, \sigma'). \tag{77}$$

It is important to decide the "missing" simplex is in position n, between the first factor σ in position $n-1$ and the second one σ' in position $n+1$. This comes from the common rule about alternating signs reflecting the orientations of faces: we must put σ and σ' in such a way they have naturally the same sign, which needs a gap between them, gap which will be filled by the composition, which is to be interpreted as if $\sigma - (\sigma \cdot \sigma') + \sigma' = 0$, the wished relation.

To finish to convince the reader about this process, let us decide to define the composition as:

$$\sigma \cdot \sigma' := \partial_{n+1} \varepsilon(n, n+1, *_n, \ldots, *_n, \sigma, \sigma'). \tag{78}$$

where this time the simplex in position $n+1$ is missing. In particular, if $\sigma = \sigma'$, we obtain:

$$\sigma \cdot \sigma := \partial_{n+1} \varepsilon(n, n+1, *_n, \ldots, *_n, \sigma, \sigma). \tag{79}$$

But there is always in this case a *trivial* solution for the Kan extension, namely $\eta_{n-1}\sigma$, for $\partial_{n-1}\eta_{n-1}\sigma = \partial_n \eta_{n-1}\sigma = \sigma$ and the other faces of $\eta_{n-1}\sigma$ are the base point. We would obtain $\sigma \cdot \sigma = *_n$, this relation is correct with the right signs: $(-1)^{n-1}\sigma + (-1)^n \sigma = *_n$; but nothing can be concluded about the composition of σ and itself, that is, with the *same sign*.

Definition 54 — Let $(K, *)$ be a pointed simplicial set satisfying the Kan extension condition. Then two n-spheres σ and σ' are *homotopic* if there exists an $(n+1)$-simplex ρ such that $\partial_i \rho = *_n$ for $0 \leq i \leq n-1$, $\partial_n \rho = \sigma$ and $\partial_{n+1}\rho = \sigma'$.

For example the discussion preceding this Definition was simply a proof any sphere is homotopic to *itself*.

Proposition 55 — *Let $(K, *)$ be a pointed simplicial set satisfyng the Kan extension condition. Let $P_n(K, *)$ be the set of n-spheres of K. Then:*

$$\pi_n(K, *) := P_n(K, *)/\sim \tag{80}$$

is the quotient of the n-sphere set byt the homotopy relation defined above, which is an equivalence relation. $\pi_0(K, *)$ is naturally identified to the connected components of K, pointed by the one containing the base point $*$. The composition of spheres define a group structure over $\pi_n(K, *)$ for $n \geq 1$, commutative for $n \geq 2$.

♣ The proof consists in copying which was explained in the topological case in Section 11.1, replacing the various topological constructions by simplicial ones, frequently using the Kan condition to guarantee such constructions are possible. See [13, Section I] for a detailed discussion. ♣

12 Simplicial fibrations.

A *fibration* is a map $p : E \to B$ between a *total space* E and a *base space* B satisfying a few properties describing more or less the total space E as a *twisted product* $F \times_\tau B$. In the simplicial context, several definitions are possible. The notion of *Kan* fibration corresponds to a situation where a simplicial homotopy lifting property is satisfied; to state this property, the elementary datum is a Kan hat in the total space and a given filling of its projection in the base space; the Kan fibration property is satisfied if it is possible to fill the Kan hat in the total space in a coherent way with respect to the given filling in the base space. This notion is the simplicial version of the notion of *Serre fibration*, a projection where the homotopy lifting property is satisfied for the maps defined on polyhedra. The reference [13] contains a detailed study of the basic facts around Kan fibrations, see [13, Chapters I and II].

We will examine with a little more details the notion of *twisted cartesian product*, corresponding to the topological notion of fibre bundle. It is a key notion in topology, and the simplicial framework is particularly favourable for several reasons. In particular the Serre spectral sequence becomes well structured in this environment, allowing us to extend it up to a *constructive* version, one of the main subjects of another lecture series of this Summer School. We give here the basic necessary definitions for the notion of twisted cartesian product.

A reasonably general situation consists in considering the case where a simplicial group G acts on the fiber space, a simplicial set F, the fiber space. As usual this means a map $\phi : F \times G \to F$ is given; source and target are simplicial sets, the first one being the product of F by the simplicial set G, underlying the simplicial group; the map ϕ is a simplicial map; furthermore each component $\phi_m : (F \times G)_m = F_m \times G_m \to F_m$ must satisfy the traditional properties of the right actions of a group on a set. We will use the shorter notation $f.g$ instead of $\phi(f, g)$. Let also B be our base space, some simplicial set.

Definition 56 — A *twisting operator* $\tau : B \to G$ is a family of maps $\{\tau_m : B_m \to G_{m-1}\}_{m\geq 1}$ satisfying the following properties.

1. $\partial_0\tau(b) = \tau(\partial_1 b)\tau(\partial_0 b)^{-1}$;
2. $\partial_i\tau(b) = \tau(\partial_{i+1}(b))$ if $i \leq 1$;
3. $\eta_i\tau(b) = \tau(\eta_{i+1}b)$;
4. $\tau(\eta_0 b) = e_m$ if $b \in G_{m+1}$, the unit element of G_m being e_m.

In particular it is not required τ is a *simplicial map*, and in fact, because of the degree -1 between source and target dimensions, this does not make sense.

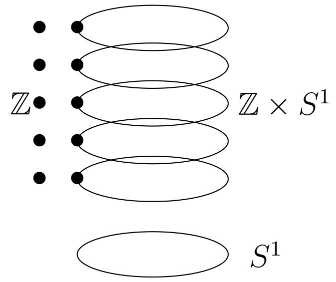
Definition 57 — If a twisting operator $\tau : B \to G$ is given, the corresponding *twisted cartesian product* $E = F \times_\tau B$ is the simplicial set defined as follows. Its set of m-simplices E_m is the same as for the non-twisted product $E_m = F_m \times B_m$; the face and degeneracy operators are also the same as for the non-twisted product with only one exception: $\partial_0(f, b) = (\partial_0 f.\, \tau(b), \partial_0 b)$.

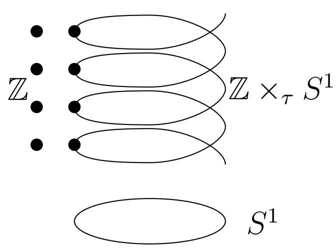
The twisting operator τ, the unique ingredient at the origin of a difference between the non-twisted product and the τ-twisted one, acts in the following way: the twisted product is constructed in a recursive way with respect to the base dimension. Let $B^{(k)}$ be the k-skeleton of B and let us suppose $F \times_\tau B^{(k)}$ is already constructed. Let σ be a $(k+1)$-simplex of B; we must describe how the product $F \times \sigma$ is to be attached to $F \times B^{(k)}$; what is above the faces $\partial_i \sigma$ for $i \geq 1$ is naturally attached; but what is above the 0-face is shifted by the translation defined by the operation of $\tau(b)$. It is not obvious such an attachment is coherent, but the various formulas of Definition 57 are exactly the relations which must be satisfied by τ for consistency. It was not obvious, starting from scratch, to guess this is a good framework for working simplicially about fibrations; this was invented (discovered ?) by Daniel Kan [12]; the previous work by Eilenberg and MacLane [5, 6] in the particular case of the fibrations relating the elements of the Eilenberg-MacLane spectra was probably determining.

12.1 The simplest example.

Let us describe in this way the exponential fibration $\mathbf{exp} : \mathbb{R} \to S^1 : t \to e^{2\pi i t}$. The base space B is S^1, a simplicial model with one vertex $*_0$ and one non-degenerate edge $\sigma = \mathrm{id}(\underline{1})$ had been described Section 4.3. The structural group G is $\mathbb{Z}$, a discrete simplicial group, see Section 4.1: any simplex set $\mathbb{Z}_n$ is simply $\mathbb{Z}$, and for every $\underline{\Delta}$-morphism $\alpha : \underline{m} \to \underline{n}$, the induced map $\alpha : \mathbb{Z}_n = \mathbb{Z} \to \mathbb{Z} = \mathbb{Z}_m$ is the identity; every simplex of dimension ≥ 1 is degenerate. The fiber space $F = \mathbb{Z}$ is again the structural group, acting on itself according to the group law. When this is the case, the fibration is called *principal*.

If we define the twisting operator $0 = \tau : S^1 \to \mathbb{Z}$ as the trivial one, $\tau(\sigma) = 0$, we obtain simply the trivial product $\mathbb{Z} \times S^1$. Because of the required compatibility with degeneracy operators, this is enough to define our twisting operator, and it it easy to verify all the required properties are satisfied.





Let us now define a non-trivial twisting operator deciding $\tau(\sigma) = 1$; again a unique coherent way to extend this definition to all the degenerate simplices of S^1. Let us carefully consider which happens for the 1-simplex $(\eta_0 n, \sigma)$, the first component being the 1-dimensional degeneracy of the integer $n \in \mathbb{Z}$. Nothing happens for the face 1: $\partial_1(\eta_0 n, \sigma) := (\partial_1 \eta_0 n, \partial_1 \sigma) = (n, *_0)$.

On the contrary, for the face 0, because of the formula:

$$\partial_0(f,b) := (\partial_0 f \cdot \tau(b), \partial_0 b) \tag{81}$$

we obtain: $\partial_0(\eta_0 n, \sigma) := (\partial_0\eta_0 n \cdot \tau(\sigma), \partial_0\sigma) = (n \cdot 1, *_0)$, but the right action of $\mathbb{Z}$ over itself, produces $\partial_0(\eta_0 n, \sigma) = (n+1, *_0)$. Because of the twisting operator, every circle in the product is in some sense "broken": it starts at the floor n, but arrives at the floor $n+1$. The realization of this twisted product therefore is homeomorphic to the real line, and the projection $\mathbb{Z} \times_\tau S^1 \to S^1$ is isomorphic to the exponential map $t \mapsto e^{2\pi i t}$.

12.2 Fibrations between $K(\pi, n)$'s.

Let us recall (see Section 9.1) $E(\pi, d)$ is the simplicial set defined by $E(\pi, d)_m = C^d(\Delta^m, \pi)$ (only *normalized* cochains) and $K(\pi, n)$ is the simplicial subset made of the *cocycles*. The maps between simplex sets to be associated to Δ-morphisms are naturally defined. A simplicial projection $p : E(\pi, d) \to K(\pi, d+1)$ associating to an m-cochain c its coboundary δc, necessarily a cocycle, is also defined. The simplicial set Δ^m is contractible, its cochain complex is acyclic and the kernel of p, the potential *fibre space*, is therefore the simplicial set $K(\pi, d)$. The base space is clearly the quotient of the total space by the fibre space (*principal* fibration), and a systematic examination of such a situation (see [13, Section 18]) shows $E(\pi, d)$ is necessarily a twisted cartesian product of the base space $K(\pi, d+1)$ by the fiber space $K(\pi, d)$.

It is not so easy to guess a corresponding twisting operator. A solution is described as follows; let $z \in Z^{d+1}(\Delta^m, \pi)$ a base m-simplex; the result $\tau(z) \in Z^d(\Delta^{m-1}, \pi)$ must be a d-cocyle of Δ^{m-1}, that is a function defined on every $(d+1)$-tuple $(i_0, \ldots, i_d)$, with values in π, and satisfying the cocycle condition; the solution $\tau(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1) - z(1, i_0+1, \ldots, i_d+1)$ works, but seems a little mysterious. The good point of view consists in considering the notion of *pseudo-section* for the studied fibration; an actual section cannot exist if the fibration is not trivial, but a pseudo-section is essentially as good as possible; see the definition of pseudo-section in [13, Section 18]. When a pseudo-section is found, a simple process produces a twisting operator; in our example, the twisting operator comes from the pseudo-section $\rho(z)(i_0, \ldots, i_d) = z(0, i_0+1, \ldots, i_d+1)$, quite natural.

The fibrations between Eilenberg-MacLane spaces are a particular case of universal fibrations associated to simplicial groups. See [13, Section 21].

12.3 Simplicial loop spaces.

A simplicial set X is *reduced* if its 0-simplex set X_0 has only one element. We have given in Section 9.2 the Kan combinatorial version GX of the loop space of X. This loop space is the fiber space of a *co-universal* fibration:

$$GX \hookrightarrow GX \times_\tau X \to X. \tag{82}$$

Only the twisting operator τ remains to be defined. The definition is simply... $\tau(\sigma) := \tau(\sigma)$ for both possible meanings of $\tau(\sigma)$; the first one is the value of the twisting operator to be defined for some simplex $\sigma \in X_{m+1}$ and the second one is the generator of GX_m corresponding to $\sigma \in X_{m+1}$, the unit element of GX_m if ever σ is 0-degenerate (see Section 9.2). The definition of the face operators for GX are exactly those which are required so that the twisting operator so defined is coherent.

It is again an example of *principal fibration*, that is the fiber space is equal to the structural group and the action $GX \times GX \to GX$ is equal to the group multiplication. This fibration is co-universal, with respect to X; in fact, let $H \hookrightarrow H \times_{\tau'} X \xrightarrow{p} X$ another *principal* fibration above X for another twisting operator $\tau' : X \to H$. Then the free group structure of GX gives you a unique group homomorphism $GX \to H$ inducing a canonical morphism between both fibrations.

If the simplicial space X is 1-reduced (only one vertex, no non-degenerate 1-simplex), then an important result by John Adams [1] allows one to compute the homology groups of GX if the initial simplicial set X is of finite type; an intermediate ingredient, the *Cobar construction*, is the key point. One of the main problems in Algebraic Topology consists in solving the analogous problem for the iterated loop spaces $G^n X$ when X is n-reduced; it is the problem of *iterating the Cobar construction*; one of the lecture series of this Summer School is devoted to this subject, organized around a *constructive* version of Algebraic Topology.

13 Homotopy groups: a quick survey.

13.1 Sphere homotopy groups.

In the same spirit as in Section 7, this section is a "cultural" presentation of the most elementary facts around the computation of homotopy groups.

The first natural question is the comparison problem between homology and homotopy groups. For simple spaces, it is common to observe the homology groups are relatively easy to be computed, and on the contrary the homotopy groups can be very hard.

The first amazing result of this sort is for the spheres: the homology groups are easily computed, it was done in these elementary notes as a consequence of Proposition 33. On the contrary, computing the homotopy groups of spheres remains a difficult subject far from being totally understood today, some popular "conjectures" meaning it is essentially impossible to achieve it. Typically the *unique* non-trivial homology group of the 4-sphere is $H_4 S^4 = \mathbb{Z}$, while the homotopy groups of the same sphere are rather chaotic:

n	4	5	6	7	8	9	10	11	12	13	$\cdots$
$\pi_n S^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12} \oplus \mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_{24}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\cdots$

where $\mathbb{Z}_k$ means $\mathbb{Z}/k\mathbb{Z}$ as it is common in algebraic topology. This dissymetry is amazing, but *Hilton's duality* gives an interesting explanation. This duality expresses on the contrary there is a good symmetry between homology and homotopy groups, at least if you choose the good point of view.

A sphere is simply (!) simple, but *another* simplicity is precisely the fact that its *homology groups* are... *simple*: an n-sphere is a *Moore space* with only *one* non-trivial homology group in dimension n, which is denoted by the relation $S^n = \mathrm{MS}(\mathbb{Z}, n)$ (MS = Moore space). Such a Moore space is *unique* up to homotopy equivalence.

Symmetrically a homotopically simple space should have only one non-trivial homotopy group; such a space does exist, it is also unique up to homotopy equivalence, it is the Eilenberg-MacLane space $K(\pi, n)$ studied in Section 9.1. For example the sphere S^4 is *the* (up to homotopy equivalence) space with $H_4 = \mathbb{Z}$ as unique non-trivial homology group. In front of this sphere, it is natural to consider the space, unique up to homotopy equivalence, having $\pi_4 = \mathbb{Z}$ as the unique non-trivial homotopy group; this space is the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$, its homotopy groups are simple, it is the very definition of this space, but the homology groups are complicated:

n	4	5	6	7	8	9	10	11	12	13	$\cdots$
$H_n K(\mathbb{Z}, 4)$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3 \oplus \mathbb{Z}$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_{60} \oplus \mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$

However the symmetry is not totally perfect: the structure of the homology groups of the Eilenberg-MacLane spaces is well understood, for a long time, due to Henri Cartan's work in the fifties [3]. On the contrary the *entire* structure of the homotopy groups of the spheres seems today out of scope for long.

13.2 Hilton's duality continued.

The right continuation of the story is along the same line. The Mayer-Vietoris exact sequence is a simple process often allowing one to compute the homology groups of $A \cup B$ when the homology groups of A, B and $A \cup B$ are known. However note the Mayer-Vietoris method *is not* an algorithm, because of a computability lack; transforming it and the other analogous exact and spectral sequences into actual algorithms is relatively recent; the article [18] gives in particular an account about this matter of computability in Algebraic Topology.

13.2.1 Serre exact sequence.

No Mayer-Vietoris exact sequence for the homotopy groups, but there is such an exact sequence for the fibrations. The respective roles of *fibrations* and *cofibra-*

tions for homology and homotopy are *exchanged*. The sophisticated tool "spectral sequence" is required to study the homology of a fibration, but for the difficult homotopy groups, the simple tool exact sequence is enough.

Proposition 58 (Serre exact sequence) — *Let $F \hookrightarrow E \to B$ be a fibration (see Section 12) with a connected base space B. Then a long exact sequence connect the homotopy groups of the components:*

$$\cdots \longleftarrow \pi_{n-1}F \longleftarrow \pi_n B \longleftarrow \pi_n E \longleftarrow \pi_n F \longleftarrow \pi_{n+1}B \longleftarrow \cdots \tag{83}$$

The morphisms $\pi_n F \to \pi_n E$ and $\pi_n E \to \pi_n B$ are naturally induced by the maps defining the fibration. The *connection morphism* $\pi_n B \to \pi_{n-1}F$ is more esoteric.

13.2.2 Homotopy groups of S^1.

Two elementary examples, already considered in these notes, are convenient to understand how to use this exact sequence. The first one is the fibration:

$$\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1 \tag{84}$$

which was used to illustrate the notion of twisting operator in Section 12.1. The situation is particularly simple but already interesting. The total space is contractible, so that all its homotopy groups are null, $\pi_n\mathbb{R} = 0$ for every $n \in \mathbb{N}$. The fiber space $\mathbb{Z}$ is a discrete space. In general, because of the role of the base point when defining the homotopy groups, $\pi_n X = \pi_n X_0$ for $n \geq 1$ if X_0 is the connected component of the base point: the other connected components cannot play any role for $n \geq 1$. Here the base point is $0 \in \mathbb{Z}$ and the connected component of 0 is $\{0\}$, a point, so that $\pi_n\mathbb{Z} = 0$ for every $n \geq 0$. On the contrary, see Section 11.1.2, $\pi_0 X$ is just the *set* of the connected components, and therefore $\pi_0\mathbb{Z} = \mathbb{Z}$ which in this particular case is a group.

The Serre exact sequence gives in particular for $n \geq 2$:

$$(\pi_{n-1}\mathbb{Z} = 0) \longleftarrow \pi_n S^1 \longleftarrow (\pi_n\mathbb{R} = 0) \tag{85}$$

and a group placed in an exact sequence between two null groups is null also, so that this proves $\pi_n S^1 = 0$ for $n \geq 2$. The circle S^1 is connected and $\pi_0 S^1 = 0$. To determine $\pi_1 S^1$, again we use the Serre exact sequence:

$$(\pi_0\mathbb{R} = 0) \longleftarrow (\pi_0\mathbb{Z} = \mathbb{Z}) \longleftarrow \pi_1 S_1 \longleftarrow (\pi_1\mathbb{R} = 0) \tag{86}$$

In an exact sequence, two consecutive groups placed between two other groups which are null are necessarily isomorphic. This proves the relation $\pi_1 S^1 = \mathbb{Z}$.

In this particular case, the connection morphism $\partial : \pi_1 S^1 \to \pi_0\mathbb{Z}$ is easy to understand. Remembering the circle can be viewed as the unit circle of the complex plane, a loop $\gamma : [0,1] \to S^1$ of the circle is also a loop $\gamma : [0,1] \to (\mathbb{C}_* := \mathbb{C} - \{0\})$, but such a loop has an *index*:

$$\alpha_\gamma := \frac{1}{2i\pi} \int_\gamma \frac{dz}{z} \tag{87}$$

which counts the number of "rounds" of the path γ around the origin of $\mathbb{C}$; this index is an integer and $\partial(\gamma) = \alpha_\gamma$.

This intrusion of the complex plane and its analysis environment could not be generalized to other connection morphisms. The key point is not at all complex analysis, it is a matter of being able to *lift* appropriate maps. The projection $\exp : \mathbb{R} \to S^1$ satisfies an essential property: if $\gamma : [0,1] \to S^1$ is a continuous map and if $x_0 \in \mathbb{R}$ satisfies $\exp(x_0) = \gamma(0)$, then there exists a unique lifting $\widehat{\gamma} : [0,1] \to \mathbb{R}$ satisfying $\widehat{\gamma}(0) = x_0$ and $\exp \circ \widehat{\gamma} = \gamma$:

$$\begin{array}{ccc} \{0\} & \xrightarrow{x_0} & \mathbb{R} \\ \downarrow & \overset{\widehat{\gamma}}{\nearrow} & \downarrow \exp \\ [0,1] & \xrightarrow[\gamma]{} & S^1 \end{array}$$

A compacity argument allows to construct the lifting $\widehat{\gamma}$ by pieces concerning only small "segments" $[\alpha, \alpha + \varepsilon]$ of the circle, where the inverse image $\exp^{-1}([\alpha, \alpha + \varepsilon])$ is homeomorphic to the *trivial* product $\mathbb{Z} \times [\alpha, \alpha + \varepsilon]$. This last property is typical of a *covering*. More generally, the homotopy lifting property, generalizing our situation, with lifting in general non-unique, allows to prove the Serre exact sequence, see for example [22, Section 7.2].

So that only one homotopy group of S^1 is non-null, and the circle is an example of Eilenberg-MacLane space: $S^1 = K(\mathbb{Z}, 1)$. Note the *standard* model of $K(\mathbb{Z}, 1)$ described in Section 9.1 is a "monster" of infinite dimension, but because of the unicity of an Eilenberg-MacLane space $K(\pi, n)$, certainly this monster[5] $K(\mathbb{Z}, 1)$ has the same homotopy type as the simple circle S^1. In fact it is not hard, using the method of round counts to construct a canonical map $|K(\mathbb{Z}, 1)| \to S^1$. The inverse map up to homotopy is the canonical inclusion $S^1 \hookrightarrow |K(\mathbb{Z}, 1)|$ the image of which is the edge [1] of $K(\mathbb{Z}, 1)$.

13.2.3 The Hopf fibration revisited.

The next example is the Hopf fibration already considered p.38:

$$S^1 \hookrightarrow S^3 \to S^2 \tag{88}$$

We know $\pi_n S^1 = 0$ for $n \geq 2$ and the Serre exact sequence in particular produces the exact sequence segment for $n \geq 3$:

$$(\pi_{n-1} S^1 = 0) \longleftarrow \pi_n S^2 \longleftarrow \pi_n S^3 \longleftarrow (\pi_n S^1 = 0) \tag{89}$$

[5] An amusing fact: this monster is the *minimal* (!) model of S^1 in Kan's theory, see [13, Section 9]

where the central morphism placed between two null groups is an isomorphism. We will see a little further a simple reason proving $\pi_3 S^3 = \mathbb{Z}$, so that $\pi_3 S^2 = \mathbb{Z}$, which was discovered by Hopf in 1935 and at this time was a surprise: it was rather expected that, as for the homology groups, $\pi_n S^2 = 0$ for $n \geq 3$; in fact we know now an infinity of groups $\pi_n S^2$ are not null. And these groups are the same for S^2 and S^3, not really intuitive!

It can be proved by relatively simple means that $\pi_i S^n$ is null for $i < n$: any continuous map $S^i \to S^n$ is homotopic to a simplicial map modulo a subdivision of the source space [7, II.7], but such a map is not surjective, and $S^n - \{*\} \cong \mathbb{R}^n$ is contractible. In particular $\pi_1 S^3 = \pi_2 S^3 = 0$. So that this segment of the Serre exact sequence:

$$(\pi_1 S^3 = 0) \longleftarrow \pi_1 S^1 \longleftarrow \pi_2 S^2 \longleftarrow (\pi_2 S^3 = 0) \tag{90}$$

proves $\pi_2 S^2 \cong \pi_1 S^1 = \mathbb{Z}$.

13.2.4 Hurewicz and Adams.

The isomorphism $\pi_2 S^2 = \mathbb{Z}$ is confirmed by the Hurewicz theorem, quite fundamental.

Theorem 59 (Hurewicz Theorem) — *Let X be a simply connected space. If $H_i X = 0$ for $1 \leq i \leq n-1$ with some $n \geq 2$, then there exists a canonical isomorphism $H_n X \cong \pi_n X$.*

♣ [13, §13] ♣

This is simply understood as follows: the first non trivial homology and homotopy groups are canonically isomorphic. Note in particular $H_0 X$ being the free $\mathbb{Z}$-group generated by the connected components of X, it is isomorphic to $\mathbb{Z}$ if X is simply connected and in particular connected. But this does not matter in the Hurewicz theorem. In the particular case $n = 1$, there is an analogous result:

Theorem 60 (Poincaré Theorem) — *Let X be a connected space. Then there is a canonical surjective map $\pi_1 X \to H_1 X$ the kernel of which is the commutator subgroup $[\pi_1 X, \pi_1 X]$.*

♣ [13, §13] ♣

In other words this produces a canonical exact sequence:

$$0 \longleftarrow H_1 X \longleftarrow \pi_1 X \hookleftarrow [\pi_1 X, \pi_1 X] \longleftarrow 0 \tag{91}$$

inducing a canonical isomorphism $H_1 X \cong \pi_1 X / [\pi_1 X, \pi_1 X]$: The first homology group is the first homotopy group *made commutative* by division by the normal subgroup generated by the commutators $[a, b] := aba^{-1}b^{-1}$, this normal subgroup is known as the *commutator subgroup.*

Let us go back to the Hurewicz theorem for the spheres: if $n \geq 2$, then $\pi_1 S^n = 0$, for $1 \leq n$ and the Hurewicz theorem can be applied. The first non-trivial homology group $H_n S^n = \mathbb{Z}$ therefore is isomorphic to the first non-trivial homotopy group $\pi_n S^n = \mathbb{Z}$.

The Hurewicz theorem, combined with the Serre spectral sequence, can be used to compute the homotopy groups as follows. More generally let X be a simply connected space the first non-trivial homotopy group being $\pi_n X = G$, some commutative group. Then it can be proved it is possible to construct a canonical fibration:

$$K(G, n-1) \hookrightarrow K(G, n-1) \times_\tau X \to X \tag{92}$$

where the fiber space is an Eilenberg-MacLane space $K(G, n-1)$: its *unique* non-trivial homotopy group is $\pi_{n-1} K(G, n-1) = G$; furthermore the twisting operator τ can be chosen in such a way the connection morphism of the Serre exact sequence $\pi_n X \to (\pi_{n-1} K(G, n-1) = G)$ is *the* isomorphism $\pi_n X \cong G$. Then working exactly as for the Hopf fibration, it was in fact a particular case of this situation, we obtain two results:

- $\pi_n(K(G, n-1) \times_\tau X) = 0$;
- For $i \geq n+1$, $\pi_i(K(G, n-1) \times_\tau X) \cong \pi_i X$.

Let us denote the total space $K(G, n-1) \times_\tau X =: X_{n+1}$. You see this total space X_{n+1} has the same homotopy groups as X except $\pi_n X_{n+1} = 0$: one says X_{n+1} is X where the $\pi_n = G$ has been *killed*. This allows the topologist to apply *again* the Hurewich theorem to X_{n+1}: we know now that $\pi_{n+1} X_{n+1} \cong H_{n+1} X_{n+1}$. If the homology groups of $K(G, n-1)$ are known and if we can apply the Serre spectral sequence 7.5.3, then we could compute the homology groups of X_{n+1} and decuce in particular of $H_{n+1} X_{n+1}$ the group $\pi_{n+1} X_{n+1} \cong \pi_{n+1} X$, and we could continue.

Let us apply for example this process to the 3-sphere S^3. The first homology group of S^3 is $H_3 S^3 = \mathbb{Z}$, which allows us to construct a fibration:

$$K(\mathbb{Z}, 2) \hookrightarrow (X_4 = K(\mathbb{Z}, 2) \times_\tau S^3) \to S^3 \tag{93}$$

It can be proved, using the homology groups of S^3 and $K(\mathbb{Z}, 2)$ that the first homology group of X_4 is $H_4 X_4 = \mathbb{Z}/2\mathbb{Z}$; therefore $\mathbb{Z}/2\mathbb{Z} = H_4 X_4 = \pi_4 X_4 = \pi_4 S^3$, which proves $\pi_4 S^3 = \mathbb{Z}/2\mathbb{Z}$.

The next step will produce a fibration:

$$K(\mathbb{Z}/2\mathbb{Z}, 3) \hookrightarrow (X_5 = K(\mathbb{Z}/2\mathbb{Z}, 3) \times_\tau X_4) \to X_4 \tag{94}$$

It can be proved, using the homology groups of X_4 and $K(\mathbb{Z}/2\mathbb{Z}, 3)$ that the first homology group of X_5 is $H_5 X_5 = \mathbb{Z}/2\mathbb{Z}$; therefore $\mathbb{Z}/2\mathbb{Z} = H_5 X_5 = \pi_5 X_5 = \pi_5 X_4 = \pi_5 S^3$, which proves $\pi_5 S^3 = \mathbb{Z}/2\mathbb{Z}$.

And so on, but the computations become more complicated; Jean-Pierre Serre, using essentially these ingredients, succeeded in 1950 in computing almost all the groups $\pi_i S^n$ for $i \leq 10$, which was an enormous progress with respect to which was known before: very few. He was rewarded for this work by a Fields Medal in 1954.

A sophisticated and powerful extension of the Hurewicz theorem is the *Adams spectral sequence*: it is a spectral sequence which essentially starts with the homology groups of X and which converges to the homotopy groups of X. But this tool is totally out of scope in these elementary notes. All the modern research works around the homotopy groups are now based upon the Adams spectral sequence, see for example [17].

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