

Walking in the Quarter Plane

Manuel Kauers (RISC)

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Doron Zeilberger (Rutgers)

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Alin Bostan (INRIA)

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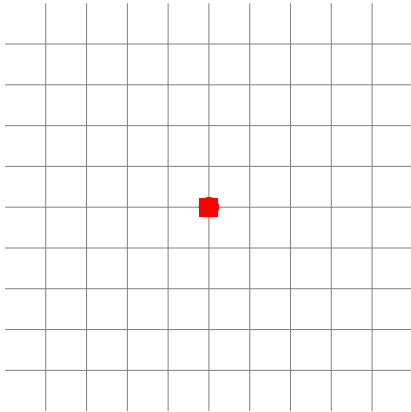
Manuel Kauers (RISC)

Doron Zeilberger (Rutgers)

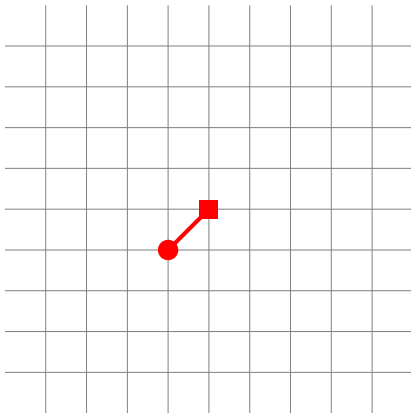
Alin Bostan (INRIA)

Christoph Koutschan (RISC)

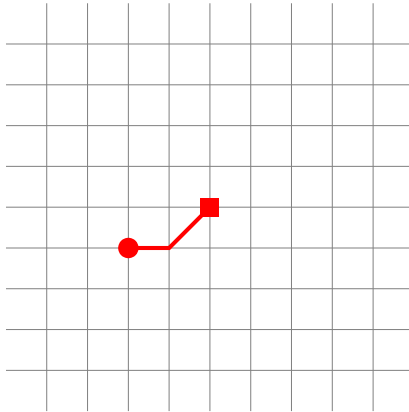
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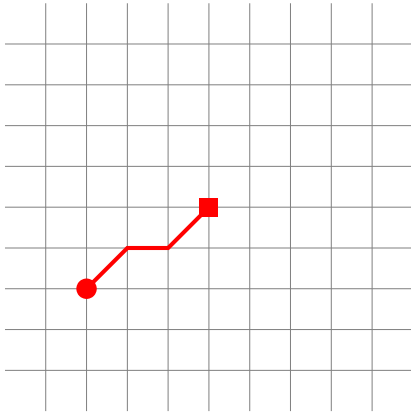
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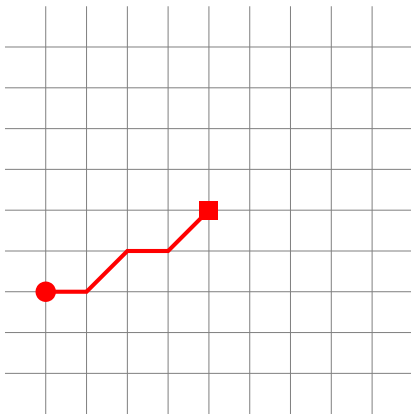
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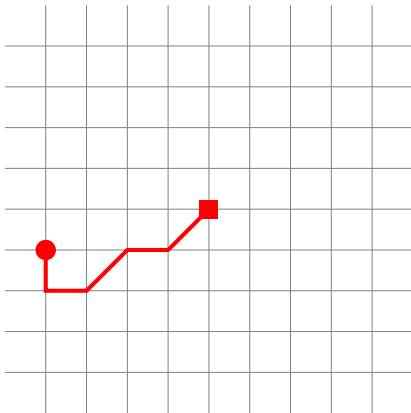
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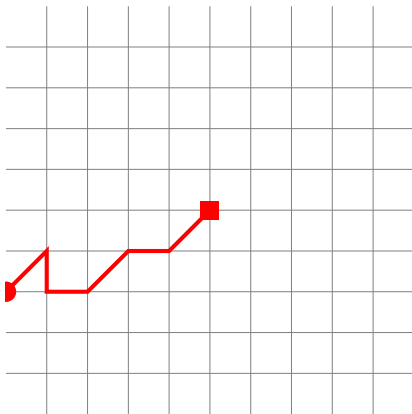
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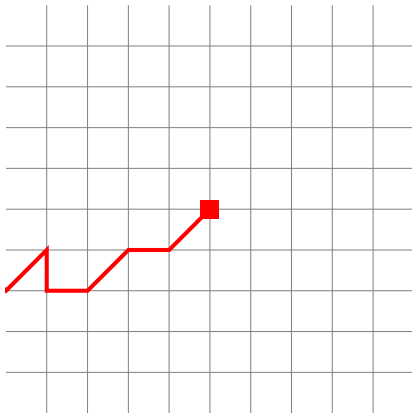
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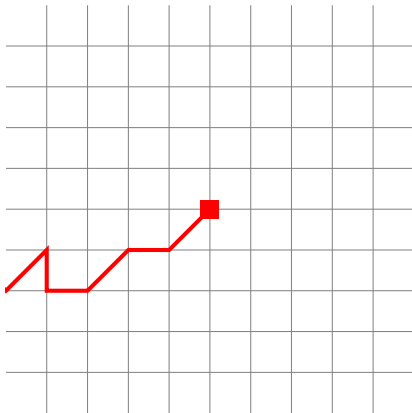
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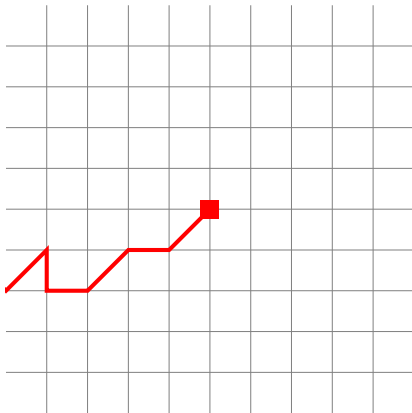
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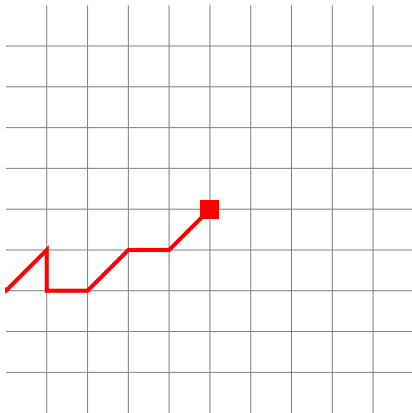
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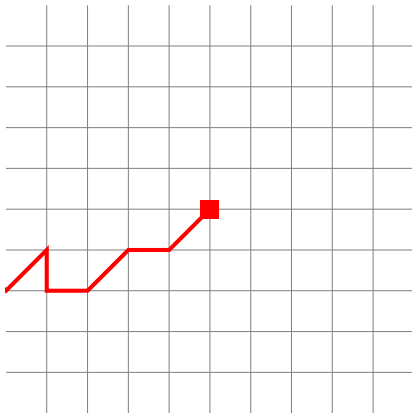
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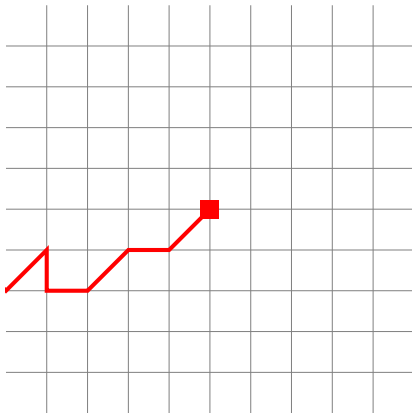
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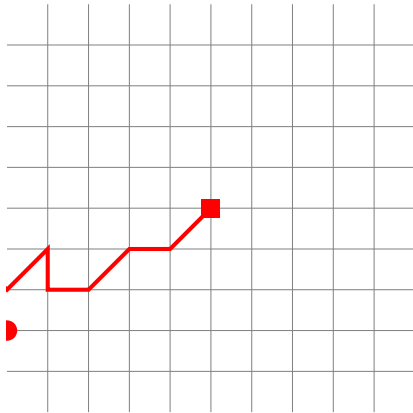
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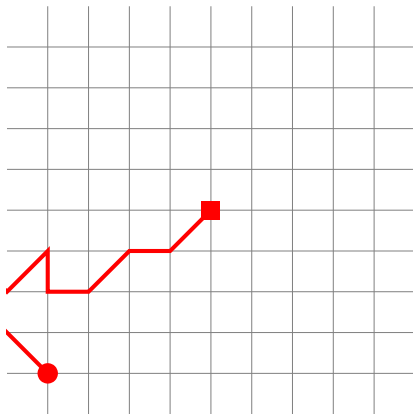
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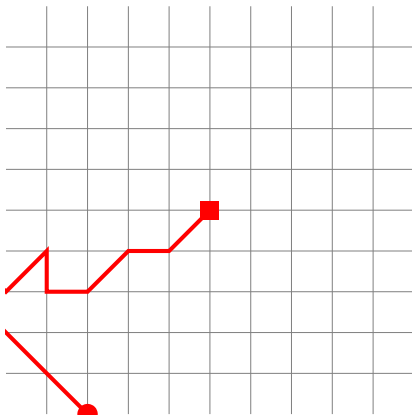
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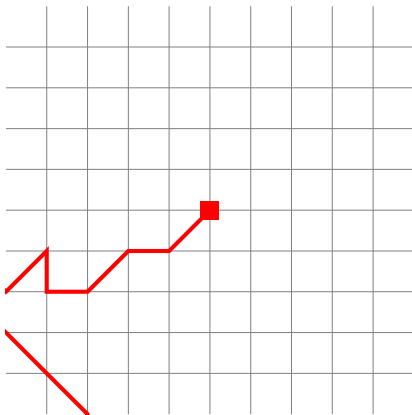
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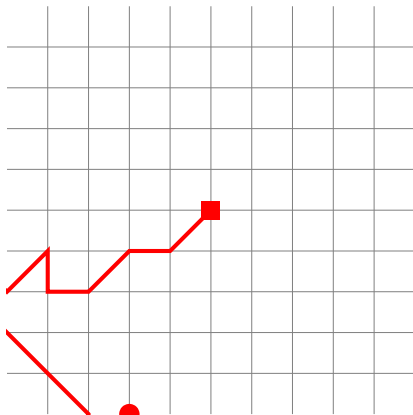
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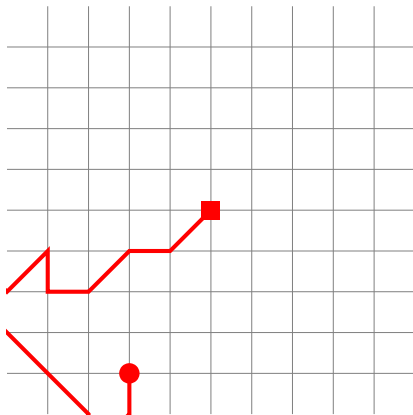
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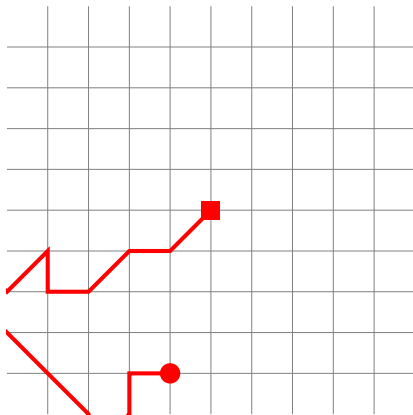
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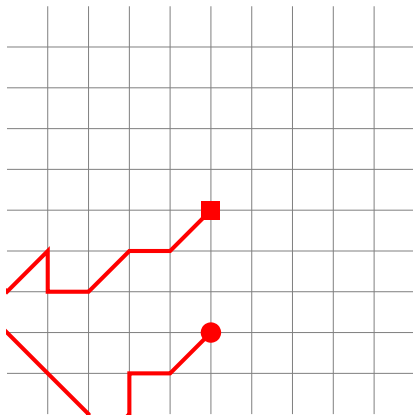
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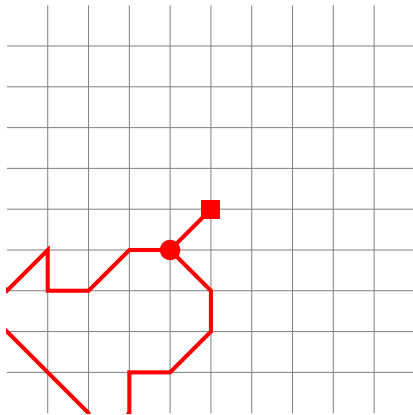
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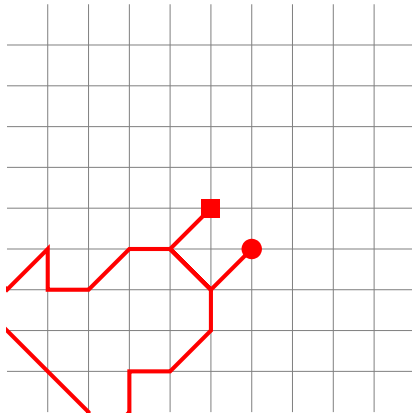
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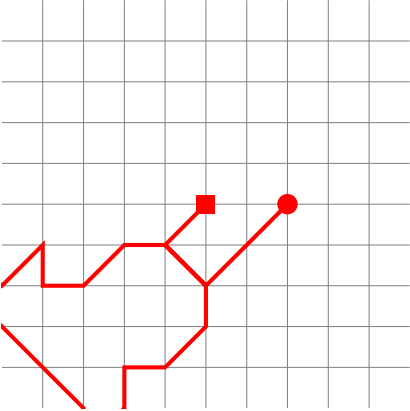
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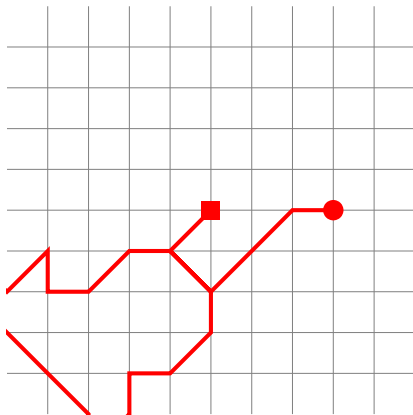
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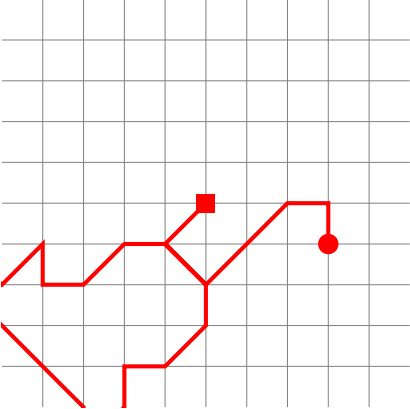
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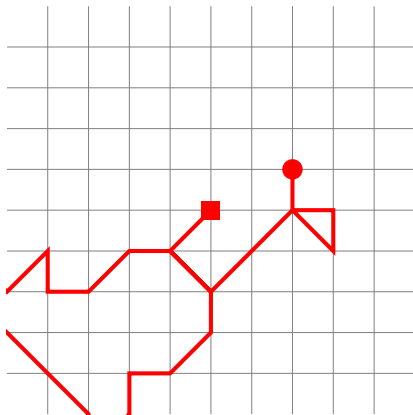
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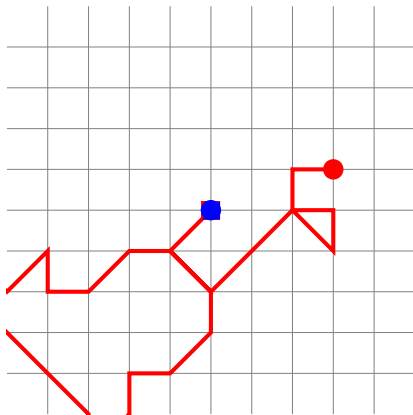
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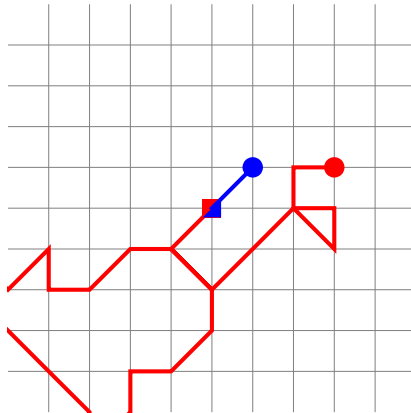
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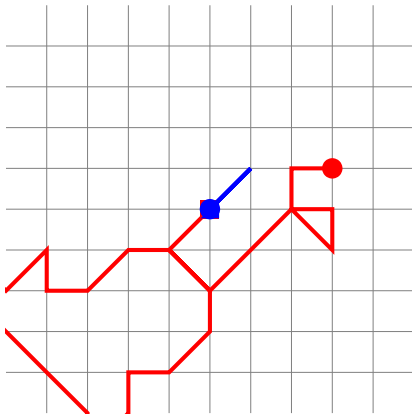
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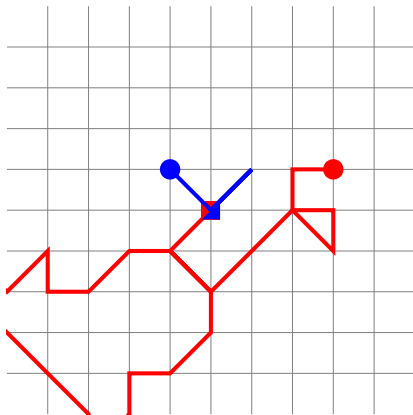
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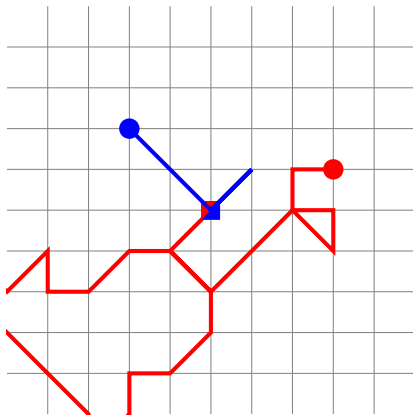
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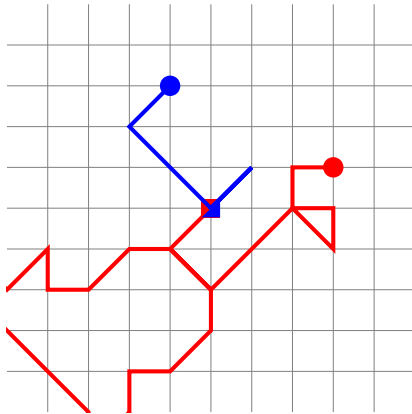
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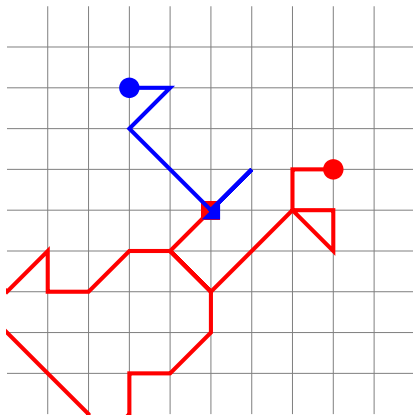
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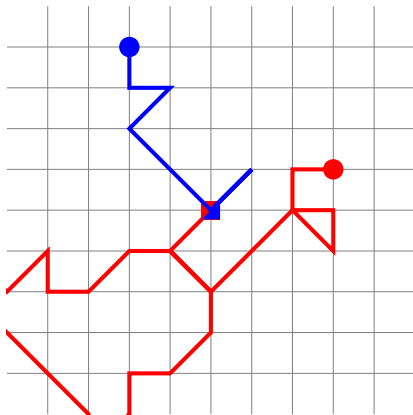
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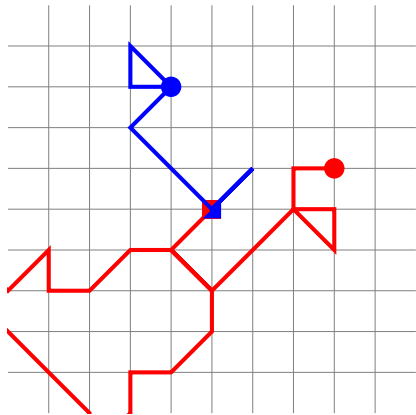
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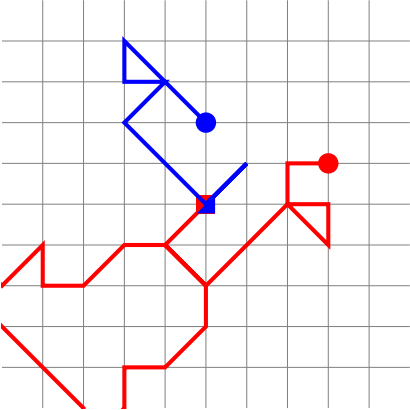
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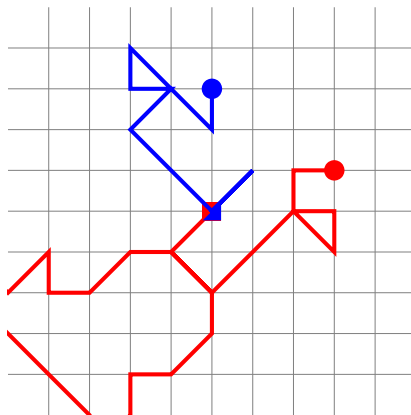
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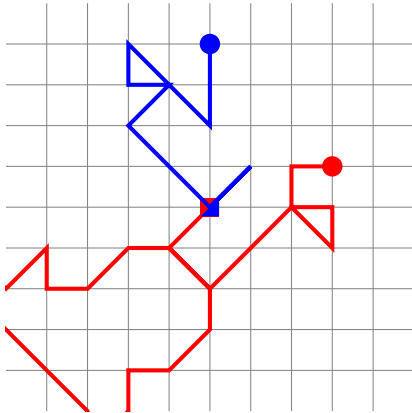
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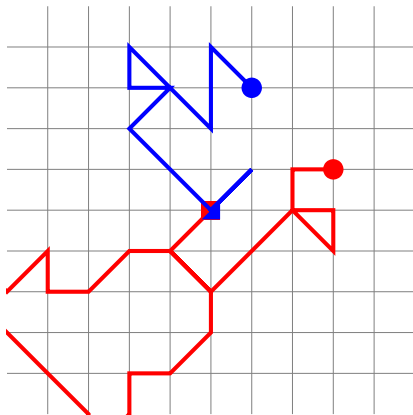
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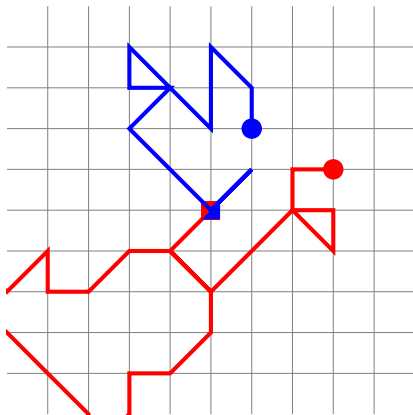
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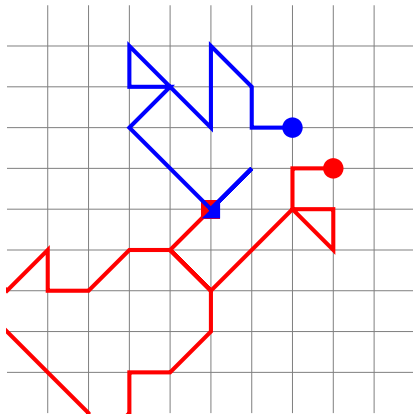
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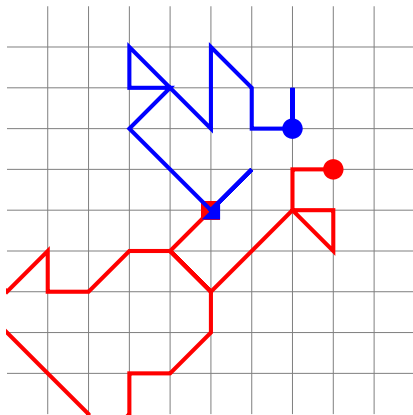
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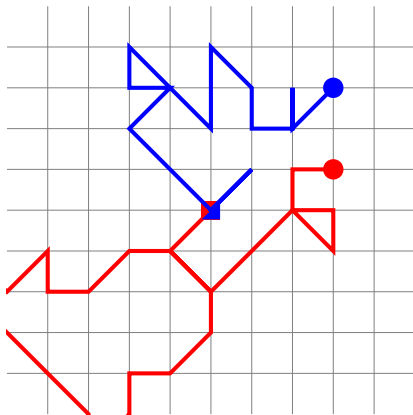
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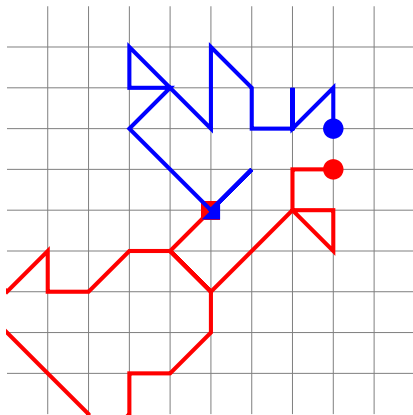
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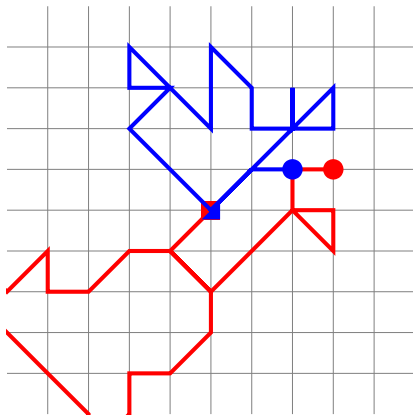
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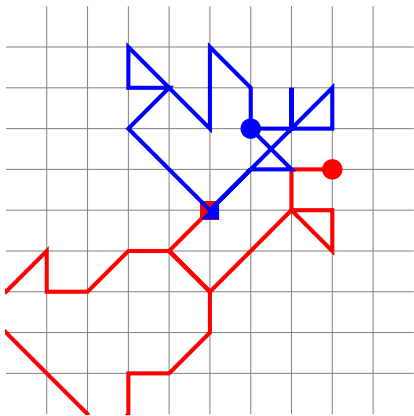
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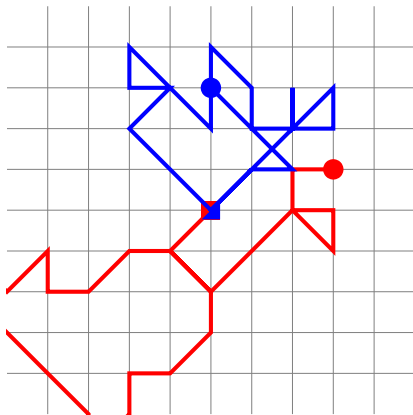
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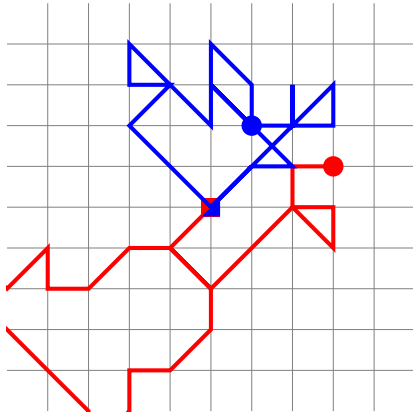
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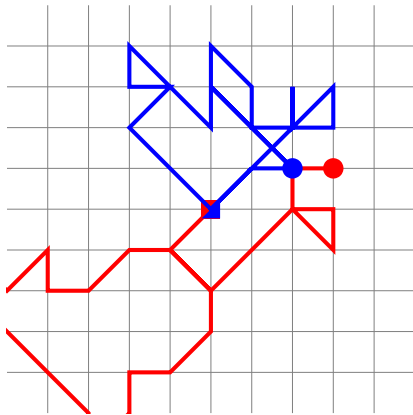
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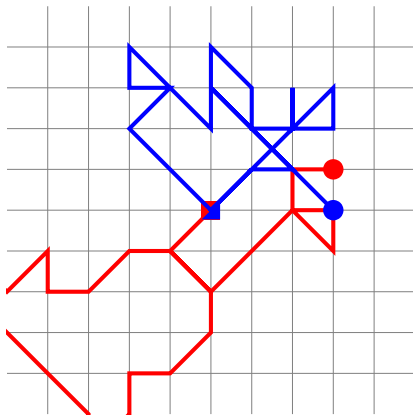
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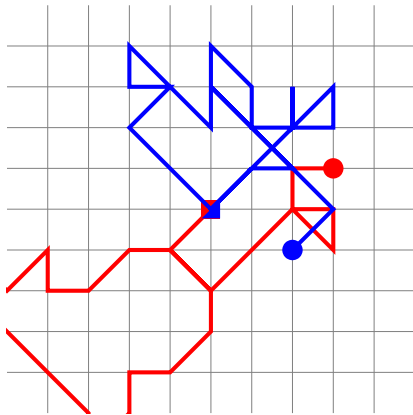
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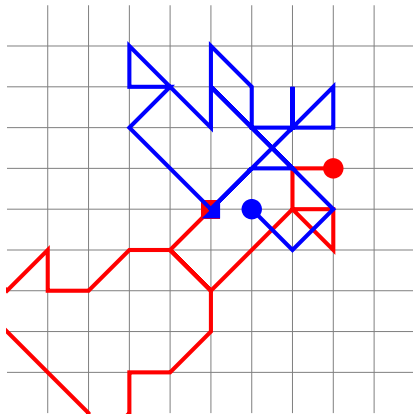
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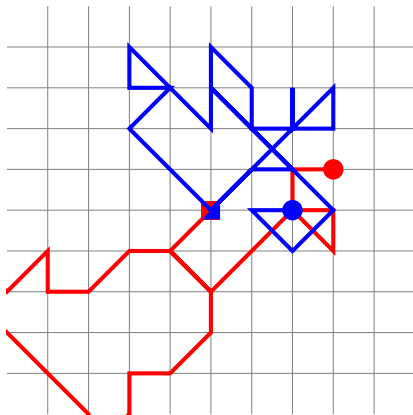
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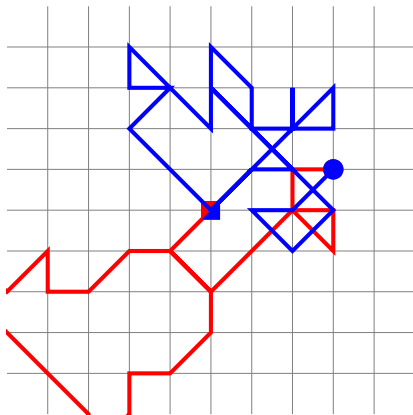
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Typical Questions

How many ways are there to walk from $(0, 0)$ to $(3, 1)$?

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How do they depend on the target point (i, j) ?

How are they influenced by restricting the area or the step set?

The Basic Recurrence

Let $f(n; i, j)$ be the number of ways to walk from $(0, 0)$ to (i, j) in exactly n steps, using only $\nearrow, \leftarrow, \downarrow$.

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Restricting the walks to the first quadrant amounts to imposing some additional boundary conditions on $f(n; i, j)$.

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$$f(n; i, j)$$

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$$f(n; i, j)x^i y^j$$

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- ▶ $\text{rat}_n(0, 1)$ is the number of walks with n steps ending somewhere on the vertical axis.

The Generating Function

For unrestricted walks,

$$\begin{aligned}f(n; i, j) &= f(n - 1; i - 1, j - 1) \\ &\quad + f(n - 1; i + 1, j) + f(n - 1; i, j + 1) \\ f(0; i, j) &= \delta_{i,j,0}\end{aligned}$$

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The generating function will be rational for any choice of allowed unit steps.

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Kreweras' Theorem

Thm. (Kreweras, 1965)

The generating function $F(t; x, y)$ of walks

- ▶ inside the first quadrant
- ▶ consisting of unit steps $\nearrow, \leftarrow, \downarrow$

is an (ugly*) algebraic function.

Moreover, $f(3n; 0, 0) = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n} (n \geq 0)$.

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The type of $F(t; x, y)$ depends crucially on the step set.

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Mishna's Classification

The type is known for all step sets of cardinality 3 (Mishna, 2007):

steps	gfun is...
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$\uparrow, \nearrow, \swarrow$	algebraic, but not rational
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(Other step sets are equivalent to those.)

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Remember: rational \Rightarrow algebraic \Rightarrow holonomic

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(Notation: $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$.)

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But how to discover a recurrence for $g(2n; 0, 0)$?

Make an Ansatz!

Make an ansatz

$$(c_0 + c_1n + c_2n^2)g(2n; 0, 0) + (c_3 + c_4n + c_5n^2)g(2n + 2; 0, 0) = 0$$

with undetermined coefficients $c_0, c_1, c_2, c_3, c_4, c_5$.

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It might fail for some $n > 5$ (although this is *veeery unlikely*.)

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And: They can be verified by an algorithm.

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A recurrence equation corresponds to an annihilating operator

$$P(n, i, j, N, I, J)g(n; i, j) = 0.$$

... and Prove Afterwards!

Known: $T := 1 + J + I^2J + I^2J^2 - IJN$ annihilates $g(n; i, j)$.

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This implies termination.

Gessel's Conjecture – 2nd Attempt

Here is a validated annihilating operator for $g(n; i, j)$:

$$\begin{aligned} & (i - 2j + n + 2)I^4J^3 + (i - 2j + n + 2)I^4J^2 \\ & - (i - 2j + n + 2)I^3NJ^2 - (3j - n - 3)I^2J^2 \\ & - (3j - n - 3)I^2J + (i + j - 1)IJN \\ & - (i + j - 1)J - (i + j - 1). \end{aligned}$$

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Here is the corresponding recurrence:

$$\begin{aligned} & (i - 2j + n + 2)g(n; i + 4, j + 3) \\ & \quad + (i - 2j + n + 2)g(n; i + 4, j + 2) \\ & - (i - 2j + n + 2)g(n + 1; i + 3, j + 2) \\ & \quad - (3j - n - 3)g(n; i + 2, j + 2) \\ & - (3j - n - 3)g(n; i + 2, j) + (i + j - 1)g(n + 1; i + 1, j + 1) \\ & - (i + j - 1)g(n; i, j + 1) - (i + j - 1)g(n; i, j) = 0. \end{aligned}$$

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This is not very useful, because of the **offsets**.

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The Chyzak-Salvy-Takayama algorithm can compute P without also computing Q .

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Idea: Apply the Chyzak-Salvy-Takayama Algorithm with i and j in place of $(I - 1)$ and $(J - 1)$ to find $P(n, N)$ with

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At this point it is routine to completing the proof of

$$g(2n; 0, 0) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}.$$

Gessel's Conjectures

Questions:

- ▶ $g(2n; 0, 0) \stackrel{?}{=} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$ for all n ?
- ▶ Is the generating function

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- ▶ $g(2n; 0, 0) \stackrel{?}{=} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$ for all n ? **Yes.**
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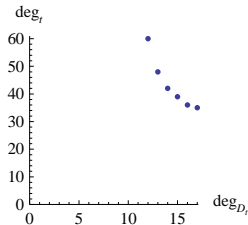
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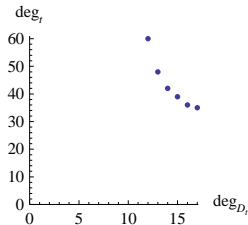


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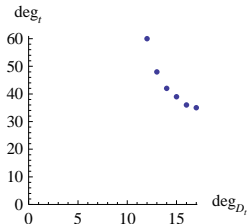
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It seems so, but $\deg_x P$ and the bit size of the integer coefficients will unreasonably large in the interpolated operator.

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Observe: If P_1, P_2 are annihilating operators of $G(t; \xi, 0)$, then so is $\text{gcd}(P_1, P_2)$.

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Observe: If P_1, P_2 are annihilating operators of $G(t; \xi, 0)$, then so is $\text{gcd}(P_1, P_2)$.

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- ▶ $\deg_t P = 96$
- ▶ $\deg_{D_t} P = 11$
- ▶ $\deg_x P = 96$
- ▶ integer coefficients with 61 decimal digits only

The operator is Fuchsian, and it has nice singularities.

A similar operator can be found for $G(t; 0, y)$.

So what...?

Holonomy

Remember: $G(t; x, y)$ satisfies the functional equation

$$G(t; x, y) = \frac{1}{1 - t(x + \frac{1}{x} + xy + \frac{1}{xy})} \\ \times \left(1 + \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0) - (1 + y)G(t; 0, y)) \right)$$

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But then there must be also a differential equation for $G(t; x, y)$...

According to estimations, it may have up to $1.5 \cdot 10^9$ terms.

And this is still not all...

Furthermore: It also seems that $G(t; x, 0)$ and $G(t; 0, y)$ are algebraic.

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Again, if this is true, then also $G(t; x, y)$ is algebraic, by the functional equation.

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Yes.

The Kernel Method

Once more:

$$\begin{aligned}(t + ty - xy + tx^2y + tx^2y^2)G(t; x, y) \\ = -xy - tG(t; 0, 0) + t(1 + y)G(t; 0, y) + tG(t; x, 0)\end{aligned}$$

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The substitution

$$\begin{aligned}y \rightarrow y(t, x) &:= \frac{x - t - tx^2 - \sqrt{(t - x + tx^2)^2 - 4t^2x^2}}{2tx^2} \\ &= \frac{1}{x}t + \frac{1+x^2}{x^2}t^2 + \frac{x^4+3x^2+1}{x^3}t^3 + \dots\end{aligned}$$

kills the left hand side and leaves us with

$$G(t; x, 0) = G(t; 0, 0) + y(t, x)x/t - (1 + y(t, x))G(t; 0, y(t, x))$$

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also kills the left hand side and leaves us with

$$(1 + y)G(t; 0, y) = G(t; 0, 0) + x(t; y)y/t - F(t; x(t, y), 0)$$

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define the series $G(t; x, 0)$ and $G(t; 0, y)$ uniquely.

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It can be checked that the *guessed* series satisfy these equations.

It follows that the guesses were correct. ■