New perspectives in algebraic systems theory

Alban Quadrat

INRIA Sophia Antipolis, APICS Project,
2004 route des lucioles, BP 93,
06902 Sophia Antipolis cedex, France.

Alban.Quadrat@sophia.inria.fr

Work in collaboration with:

J.-F. Pommaret (ENPC, France),
F. Chyzak (INRIA Rocquencourt, France),
D. Robertz (RWTH Aachen, Germany),
A. Fabiańska (RWTH Aachen, Germany),
T. Cluzeau (ENSIL, France).

MAP Conference, ICTP
Trieste, Italy (25-29/08/08)

Outline

The purpose of this talk is to develop the following 3 ideas:

1. A large class of linear functional systems can be studied by means of a non-commutative polynomial approach over skew polynomial rings and Ore algebras of functional operators.

   Non-commutative Gröbner bases $\Rightarrow$ constructive approach.

2. Algebraic analysis is a natural mathematical framework for the intrinsic study of linear systems theory (module theory).

3. Constructive homological algebra allows us to develop algorithms and symbolic packages dedicated to the study of the structural properties of multidimensional linear systems.

Matrices of differential operators

- Newton: Fluxion calculus (1666) ("dot-age")

$$\left\{\begin{array}{l} \ddot{x}_1(t) + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \ddot{x}_2(t) + \alpha\, x_2(t) - \alpha\, u(t) = 0, \end{array}\right. \qquad \alpha = g/l.$$

- Leibniz: Infinitesimal calculus (1676) ("d-ism")

$$\left\{\begin{array}{l} \frac{d^2 x_1(t)}{dt^2} + \alpha\, x_1(t) - \alpha\, u(t) = 0, \\ \frac{d^2 x_2(t)}{dt^2} + \alpha\, x_2(t) - \alpha\, u(t) = 0. \end{array}\right.$$

- Boole: Operational calculus (1859-60)

$$\left(\begin{array}{ccc} \frac{d^2}{dt^2} + \alpha & 0 & -\alpha \\ 0 & \frac{d^2}{dt^2} + \alpha & -\alpha \end{array}\right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ u(t) \end{array}\right) = 0.$$

$\Rightarrow$ Ring of differential operators $D = \mathbb{Q}(\alpha)\left[\frac{d}{dt}\right]$:

$$\sum_{i=0}^{n} a_i \left(\frac{d}{dt}\right)^i \in D, \quad a_i \in \mathbb{Q}(g, l), \quad \left(\frac{d}{dt}\right)^i = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt} = \frac{d^i}{dt^i}.$$

Functional operators

- Differential operators: $\left(\sum_{j=0}^{m} b_j(t)\, \frac{d^j}{dt^j}\right) \left(\sum_{i=0}^{n} a_i(t)\, \frac{d^i}{dt^i}\right)$

$$\frac{d}{dt}(a\,y) = a\,\frac{d}{dt}\,y + \left(\frac{da}{dt}\right)\,y \quad \Rightarrow \quad \frac{d}{dt}\,a\,\cdot = a\,\frac{d}{dt}\,\cdot + \frac{da}{dt}\,\cdot$$

- Shift operators: $\delta\, a(t) = a(t-h), \quad \sigma\, a_n = a_{n+1}$.

$$\delta\,(a(t)\,y(t)) = a(t-h)\,y(t-h) = \delta\,a\,\delta\,y \quad \Rightarrow \quad \delta\,a\,\cdot = (\delta\,a)\,\delta\,\cdot$$

$$\sigma\,(a_n\,y_n) = a_{n+1}\,y_{n+1} = \sigma\,a\,\sigma\,y \quad \Rightarrow \quad \sigma\,a\,\cdot = (\sigma\,a)\,\sigma\,\cdot$$

- Difference operators: $\Delta\, a(x) = a(x+1) - a(x)$.

$$\Delta\,a(x)\,\cdot = a(x+1)\,\Delta\,\cdot + (\Delta\,a)\,\cdot$$

- Divided difference operators: $d_{x_0} a(x) = \frac{a(x) - a(x_0)}{x - x_0}$.

$$d_{x_0}\,a(x)\,\cdot = a(x_0)\,d_{x_0}\,\cdot + (d_{x_0}\,a)\,\cdot$$

- q-difference, q-shift, q-dilation, Frobenius, Euler operators. . .

Skew polynomial rings (Ore, 1933)

- Definition: A skew polynomial ring $A\,[\partial;\alpha,\beta]$ is a non-commutative polynomial ring in ∂ with coefficients in A satisfying

$$\forall\, a \in A, \quad \boxed{\partial\, a = \alpha(a)\,\partial + \beta(a)}$$

where $\alpha : A \longrightarrow A$ and $\beta : A \longrightarrow A$ are such that:

$$\begin{cases} \alpha(1) = 1, \\ \alpha(a+b) = \alpha(a) + \alpha(b), \\ \alpha(a\,b) = \alpha(a)\,\alpha(b), \end{cases} \qquad \begin{cases} \beta(a+b) = \beta(a) + \beta(b), \\ \beta(a\,b) = \alpha(a)\,\beta(b) + \beta(a)\,b. \end{cases}$$

- $P \in A\,[\partial;\alpha,\beta]$ has a unique form $P = \sum_{i=0}^{n} a_i\,\partial^i,\ a_i \in A$.
    - Ring of differential operators: $A\left[\partial;\mathrm{id},\frac{d}{dt}\right]$.
    - Ring of shift operators: $A[\partial;\delta,0]$, $A[\partial;\sigma,0]$.
    - Ring of difference operators: $A[\partial;\tau,\tau-\mathrm{id}]$, $\tau a(x) = a(x+1)$.

Ore algebras (Chyzak-Salvy, 1996)

• We can iterate skew polynomial rings to get Ore extensions:

$$A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]$$

• Definition: An Ore extension $A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_n; \alpha_n, \beta_n]$ is called an Ore algebras if the ∂_i's commute, i.e., if we have

$$1 \leq j < i \leq m, \quad \alpha_i(\partial_j) = \partial_j, \quad \beta_i(\partial_j) = 0,$$

and the $\alpha_{i\,|_A}$'s and $\beta_{j\,|_A}$'s commute for $i \neq j$.

- Ring of differential operators: $A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$.
- Ring of differential delay operators: $A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right] [\partial_2; \delta, 0]$.
- Ring of shift operators: $A[\partial_1; \sigma_1, 0] \ldots [\partial_n; \sigma_n, 0]$.

Matrix of functional operators

- The wind tunnel model (Manitius, IEEE TAC 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0. \end{array}\right. \qquad (\star)$$

- We introduce the commutative Ore algebra:

$$D = \mathbb{Q}(a, k, \omega, \zeta) \left[\partial_1; \mathrm{id}, \frac{d}{dt}\right] [\partial_2; \delta, 0].$$

- The system $(\star)$ can be rewritten as:

$$\left(\begin{array}{cccc} \partial_1 + a & -k\, a\, \partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{array}\right) \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{array}\right) = 0.$$

Matrix of functional operators

- Linearization of the Navier-Stokes $\sim$ a parabolic Poiseuille profile

$$\begin{cases} \partial_t\, u_1 + 4\, y\, (1-y)\, \partial_x\, u_1 - 4\, (2\, y - 1)\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_1 + \partial_x\, p = 0, \\ \partial_t\, u_2 + 4\, y\, (1-y)\, \partial_x\, u_2 - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2)\, u_2 + \partial_y\, p = 0, \qquad (*) \\ \partial_x\, u_1 + \partial_y\, u_2 = 0. \qquad \text{(e.g., Vazquez-Krstic, IEEE TAC 07)} \end{cases}$$

- Let us introduce the so-called Weyl algebra ($\partial_x\, x = x\, \partial_x + 1$):

$$D = \mathbb{Q}(Re)[t, x, y] \left[\partial_t; \mathrm{id}, \frac{\partial}{\partial t}\right] \left[\partial_x; \mathrm{id}, \frac{\partial}{\partial x}\right] \left[\partial_y; \mathrm{id}, \frac{\partial}{\partial y}\right].$$

- The system $(*)$ is defined by the matrix of PD operators:

$$\begin{pmatrix} \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & -4\, (2\, y - 1) & \partial_x \\ 0 & \partial_t + 4\, y\, (1-y)\, \partial_x - \frac{1}{Re}\, (\partial_x^2 + \partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}.$$

Non-commutative Gröbner bases

- Let $D = A[\partial_1; \alpha_1, \beta_1] \ldots [\partial_m; \alpha_m, \beta_m]$ be an Ore algebra.

- Theorem: (Kredel, 93) Let $A = k[x_1, \ldots, x_n]$ a commutative polynomial ring ($k = \mathbb{Q}$, $\mathbb{F}_p$) and D an Ore algebra satisfying

$$\alpha_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \beta_i(x_j) = c_{ij},$$

for certain $0 \neq a_{ij} \in k$, $b_{ij} \in k$, $c_{ij} \in A$ and $\deg(c_{ij}) \leq 1$. Then, a non-commutative version of Buchberger's algorithm terminates for any term order and its result is a Gröbner basis.

- Implementation in the Maple package Ore_algebra (Chyzak) (Singular, Macaulay 2, NCAlgebra, JanetOre...).

- Gröbner bases can be used to effectively compute over D.

Algebraic analysis

- Let D be an Ore algebra and $R \in D^{q \times p}$.
- Let us consider the left D-morphism (i.e., D-linear application):

$$\begin{array}{ccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1 \ldots \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$\boxed{M = D^{1\times p}/\mathrm{im}_D(.R) = D^{1\times p}/(D^{1\times q}\, R).}$$

- M is formed by the equivalence classes $\pi(\mu)$ of $\mu \in D^{1\times p}$ for the equivalence relation $\sim$ on $D^{1\times p}$:

$$\mu_1 \sim \mu_2 \;\Leftrightarrow\; \exists\, \lambda \in D^{1\times q} : \; \mu_1 = \mu_2 + \lambda\, R \;\Leftrightarrow\; \mu_1 - \mu_2 \in D^{1\times q}\, R.$$

1. Number theory: $\mathbb{C} = \mathbb{R}[x]/(x^2+1)$, $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2+5)$.
2. Algebraic geometry: $\mathbb{C}[x,y]/(x^2+y^2-1, x-y)$.

Linear systems of equations

- $M = D^{1\times p}/(D^{1\times q}\, R)$ can be defined by generators and relations:
- Let $\{f_k\}_{k=1,\ldots,p}$ the standard basis of $D^{1\times p}$ ($f_k = (0\ldots 1 \ldots 0)$).
- Let $\pi : D^{1\times p} \longrightarrow M$ be the D-morphism sending μ to $\pi(\mu)$.

$$\forall\, m \in M, \ \exists\, \mu = (\mu_1 \ldots \mu_p) \in D^{1\times p} : \ m = \pi(\mu) = \sum_{k=1}^{p} \mu_k\, \pi(f_k),$$

$\Rightarrow \{y_k = \pi(f_k)\}_{k=1,\ldots,p}$ is a family of generators of M.

$$\pi((R_{l1} \ldots R_{lp})) = \pi\left(\sum_{k=1}^{p} R_{lk}\, f_k\right) = \sum_{k=1}^{p} R_{lk}\, y_k = 0, \quad l = 1, \ldots, q,$$

$\Rightarrow y = (y_1 \ldots y_p)^T$ satisfies the relation $R\, y = 0$.

Duality modules — systems

- Let $\mathcal{F}$ be a left D-module

$$\forall\, f_1,\, f_2 \in \mathcal{F}, \quad \forall\, d_1,\, d_2 \in D: \; d_1\, f_1 + d_2\, f_2 \in \mathcal{F},$$

and $\hom_D(M, \mathcal{F})$ the abelian group:

$$\hom_D(M, \mathcal{F}) = \{f : M \longrightarrow \mathcal{F} \mid f(d_1\, m_1 + d_2\, m_2) = d_1\, f(m_1) + d_2\, f(m_2)\}.$$

- Theorem (Malgrange):

$$\boxed{\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\, \eta = 0\}}$$

- $\hom_D(M, \mathcal{F})$ intrinsically characterizes the system $\ker_{\mathcal{F}}(R.)$ as it does not depend on the embedding of $\ker_{\mathcal{F}}(R.)$ into $\mathcal{F}^p$.

- We assume that $\mathcal{F}$ is an injective cogenerator left D-module

$\Rightarrow$ study of the system $\hom_D(M, \mathcal{F})$ by means of properties of M.

Module theory

- Definition: 1. M is free if $\exists\, r \in \mathbb{Z}_+$ such that $M \cong D^r$.

2. M is projective if $\exists\, r \in \mathbb{Z}_+$ and a D-module P such that:

$$M \oplus P \cong D^r.$$

3. M is reflexive if $\varepsilon : M \longrightarrow \hom_D(\hom_D(M, D), D)$ is an isomorphism, where:

$$\varepsilon(m)(f) = f(m), \quad \forall\, m \in M, \quad f \in \hom_D(M, D).$$

4. M is torsion-free if:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

5. M is torsion if $t(M) = M$.

Classification of modules

- Theorem: 1. We have the following implications:

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. If D is a principal domain (e.g., $K\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{free}.$$

3. If D is a hereditary ring (e.g., $\mathbb{Q}[t]\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$), then:

$$\text{torsion-free} = \text{projective}.$$

4. If $D = k[x_1, \ldots, x_n]$ and k a field, then:

$$\text{projective} = \text{free} \quad \text{(Quillen-Suslin theorem)}.$$

4. If $D = A_n(k)$ or $B_n(k)$, k is a field of characteristic 0, then

$$\text{projective} = \text{free} \quad \text{(Stafford theorem)},$$

for modules of rank at least 2.

Dictionary systems — modules

Module M	**Structural properties** $\ker_{\mathcal{F}}(R.)$	**Stabilization problems** **Optimal control**
Torsion	Autonomous system Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Parametrizability, π-freeness	Variational problem without constraints (Euler-Lagrange equations)
Projective	Bézout identities, Internal stabilizability	Computation of Lagrange parameters without integration Existence of a parametrization all stabilizing controllers
Free	Flatness, Poles placement, Doubly coprime factorization	Youla-Kučera parametrization Optimal controller

Oberst, Pommaret, Fliess, Q., . . .

Involutions, adjoints and dual systems

- Definition: A linear map $\theta : D \longrightarrow D$ is an involution of D if:

$$\forall\, P,\, Q \in D: \;\; \theta(P\,Q) = \theta(Q)\,\theta(P), \quad \theta^2 = \mathrm{id}.$$

- Example: 1. If D is a commutative ring, then $\theta = \mathrm{id}$.

2. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{\partial}{\partial x_1}\right] \ldots \left[\partial_n; \mathrm{id}, \frac{\partial}{\partial x_n}\right]$ is:

$$\forall\, a \in A, \quad \theta(a(x)) = a(x), \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \ldots, n.$$

3. An involution of $D = A\left[\partial_1; \mathrm{id}, \frac{d}{dt}\right]\left[\partial_2; \delta, 0\right]$ is defined by:

$$\forall\, a \in A, \quad \theta(a(t)) = a(-t), \quad \theta(\partial_i) = \partial_i, \quad i = 1, 2.$$

- The adjoint of $R \in D^{q \times p}$ is defined by $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.
- $N = D^{1 \times q}/(D^{1 \times p}\,\theta(R))$ is called the transposed of M.

Module M	**Homological algebra**	$\mathcal{F}$ **injective cogenerator**
with torsion	$t(M) \cong \text{ext}_D^1(N, D)$	$\emptyset$
torsion-free	$\text{ext}_D^1(N, D) = 0$	$\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{l_1}$
reflexive	$\text{ext}_D^i(N, D) = 0$ $i = 1, 2$	$\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$
projective	$\text{ext}_D^i(N, D) = 0$ $1 \leq i \leq n$	$\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{l_1}$ $\ker_{\mathcal{F}}(Q_1.) = Q_2\,\mathcal{F}^{l_2}$ $\ldots$ $\ker_{\mathcal{F}}(Q_{n-1}.) = Q_n\,\mathcal{F}^{l_n}$
free	Quillen-Suslin theorem Stafford's therorem	$\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^{l}$ $\exists\ T \in D^{l \times p} :\ \ T\,Q = I_l$

Extension functor $\mathrm{ext}_D^1(\,\cdot\,, D)$

- Parametrizability: $R\,y = 0 \overset{?}{\Longleftrightarrow} \exists\, Q \in D^{p \times m} : \; y = Q\,z.$

$$4. \quad \theta(P)\,z = y \quad \Longrightarrow \quad R\,y = 0 \quad 1.$$

$$\Uparrow \text{ involution } \theta \Uparrow \qquad \Downarrow \text{ involution } \theta \Downarrow$$

$$3. \quad 0 = P\,\mu \quad \overset{\mathrm{Gb}}{\Longleftrightarrow} \quad \theta(R)\,\lambda = \mu \quad 2.$$

$$P \circ \theta(R) = 0 \quad \Longrightarrow \quad \theta(P \circ \theta(R)) = \theta^2(R) \circ \theta(P) = R \circ \theta(P) = 0.$$

$$5. \; \theta(P)\,z = y \quad \overset{\mathrm{Gb}}{\Longleftrightarrow} \quad R'\,y = 0, \quad R' \in D^{q' \times p}.$$

$$\boxed{\mathrm{ext}_D^1(N, D) \cong (D^{1 \times q'}\,R')/(D^{1 \times q}\,R)}$$

6. Using Gb, we can test whether or not $\mathrm{ext}_D^1(N, D) = 0$

Wind tunnel model (Manitius, IEEE TAC 84)

1. The w.t.m. is defined by the under-determined system:

$$\begin{pmatrix} \partial_1 + a & -k\,a\,\partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{ \begin{array}{l} (\partial_1 + a)\;\lambda_1 = \mu_1, \\ -k\,a\,\partial_2\,\lambda_1 + \partial_1\,\lambda_2 + \omega^2\,\lambda_3 = \mu_2, \\ -\lambda_2 + (\partial_1 + 2\,\zeta\,\omega)\;\lambda_3 = \mu_3, \\ -\omega^2\,\lambda_3 = \mu_4. \end{array} \right. \qquad (2)$$

(2) is over-determined $\stackrel{\text{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Wind tunnel model (Manitius, IEEE TAC 84)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$\omega^2\, k\, a\, \partial_2\, \mu_1 + \omega^2\, (\partial_1 - a)\, \mu_2 + \omega^2\, \left(\partial_1^2 + a\, \partial_1\right)\, \mu_3$$
$$+ \left(\partial_1^3 + 2\, \zeta\, \omega\, \partial_1^2 + a\, \partial_1^2 + \omega^2\, \partial_1 + 2\, a\, \zeta\, \omega\, \partial_1 + a\, \omega^2\right)\, \mu_4 = 0.$$

4. We consider the over-determined system $P^T\, z = y$.

$$\left\{\begin{array}{l} \omega^2\, k\, a\, \partial_2\, z = x_1, \\ \omega^2\, (\partial_1 - a)\, z = x_2, \\ \omega^2\, (\partial_1^2 + a\, \partial_1)\, z = x_3, \\ (\partial^3 + (2\, \zeta\, \omega + a)\, \partial_1^2 + (\omega^2 + 2\, a\, \omega\, \zeta)\, \partial_1 + a\, \omega)\, z = u. \end{array}\right. \qquad (4)$$

5. The compatibility conditions of $P^T\, z = y$ are exactly generated by $R\, y = 0$ and (4) is a parametrization of the w.t.m.

Moving tank (Petit, Rouchon, IEEE TAC 02)

1. The model of a moving tank is defined by:

$$\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = 0.$$

2. We compute $\theta(R) = R^T$ and define $\theta(R)\,\lambda = \mu$:

$$\left\{ \begin{array}{l} \partial_1\,\lambda_1 + \partial_1\,\partial_2^2\,\lambda_2 = \mu_1, \\ -\partial_1\,\partial_2^2\,\lambda_1 - \partial_1\,\lambda_2 = \mu_2, \\ a\,\partial_1^2\,\partial_2\,\lambda_1 + a\,\partial_1^2\,\partial_2\,\lambda_2 = \mu_3. \end{array} \right. \qquad (2)$$

(2) is over-determined $\stackrel{\text{Gb}}{\Longrightarrow}$ compatibility conditions $P\,\mu = 0$.

Moving tank (Petit, Rouchon, IEEE TAC 02)

3. We obtain the compatibility condition $P\,\mu = 0$:

$$a\,\partial_1\,\partial_2\,\mu_1 - a\,\partial_1\,\partial_2\,\mu_2 - (1+\partial_2^2)\,\mu_3 = 0.$$

4. We consider the over-determined system $P^T\,z = y$.

$$\left\{\begin{array}{l} a\,\partial_1\,\partial_2\,z = y_1, \\ -a\,\partial_1\,\partial_2\,z = y_2, \\ -(1+\partial_2^2)\,z = y_3. \end{array}\right. \qquad (4)$$

5. The compatibility conditions of $P^T\,z = y$ are $R'\,y = 0$:

$$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & (1+\partial_2^2) & -a\,\partial_1\,\partial_2 \end{array}\right) \left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right) = 0.$$

Moving tank (Petit, Rouchon, IEEE TAC 02)

$$t(M) \cong \mathrm{ext}^1_D(N, D) \cong$$

$$\left(D^{1\times 2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\partial_2^2 & -a\,\partial_1\,\partial_2 \end{pmatrix}\right) / \left(D^{1\times 2}\begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & a\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & a\,\partial_1^2\,\partial_2 \end{pmatrix}\right)$$

$$\left\{\begin{array}{l} y_1 + y_2 = z_1, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{array}\right. \quad \overset{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_1 = 0.$$

$$\left\{\begin{array}{l} (1+\partial_2^2)\,y_2 - a\,\partial_1\,\partial_2\,y_3 = z_2, \\ \partial_1\,y_1 - \partial_1\,\partial_2^2\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \\ \partial_1\,\partial_2^2\,y_1 - \partial_1\,y_2 + a\,\partial_1^2\,\partial_2\,y_3 = 0, \end{array}\right. \quad \overset{\mathrm{Gb}}{\Longrightarrow} \quad \partial_1\,(\partial_2^2 - 1)\,z_2 = 0.$$

$\Rightarrow$ $z_1(t)$ and $z_2(t)$ are autonomous elements.

Examples

- 2D Stokes equations:

$$\begin{pmatrix} -\nu\,(\partial_x^2+\partial_y^2) & 0 & \partial_x \\ 0 & -\nu\,(\partial_x^2+\partial_y^2) & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$$

$$\Rightarrow \left\{ \begin{array}{l} (\partial_x^2+\partial_y^2)^2\, u = 0, \\ (\partial_x^2+\partial_y^2)^2\, v = 0, \\ (\partial_x^2+\partial_y^2)\, p = 0. \end{array} \right. \qquad \text{torsion module}$$

- Moving tank (Petit, Rouchon, IEEE TAC 02):

$$\left\{ \begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-2\,h) + \alpha\,\ddot{y}_3(t-h) = 0, \\ \dot{y}_1(t-2\,h) - \dot{y}_2(t) + \alpha\,\ddot{y}_3(t-h) = 0, \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} z_1(t) = y_1(t) + y_2(t), \\ z_2(t) = y_2(t) + y_2(t-2\,h) - a\,\dot{y}_3(t-h), \\ \frac{d}{dt}\,(1-\delta^2)\, z_i(t) = 0, \;\; i = 1,\, 2. \end{array} \right. \qquad \text{module with torsion}$$

Examples: torsion-free modules

- Wind tunnel model (Manitius, IEEE TAC 84):

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = \omega^2\, k\, a\, z(t-h), \\ x_2(t) = \omega^2\, \dot{z}(t) - a\,\omega^2\, z(t), \\ x_3(t) = \omega^2\, \ddot{z}(t) + \omega^2\, a\, \dot{z}(t), \\ u(t) = z(t)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{z}(t) + (\omega^2 + 2\, a\,\omega\,\zeta)\, \dot{z}(t) + a\,\omega\, z(t). \end{cases}$$

$\Rightarrow$ motion planning and tracking (Fliess et al).

- 2D stress tensor (elasticity theory):

$$\begin{cases} \partial_x\, \sigma^{11} + \partial_y\, \sigma^{12} = 0, \\ \partial_x\, \sigma^{12} + \partial_y\, \sigma^{22} = 0, \end{cases} \Leftrightarrow \begin{cases} \sigma^{11} = \partial_y^2\, \lambda, \\ \sigma^{12} = -\partial_x\, \partial_y\, \lambda, \\ \sigma^{22} = \partial_x^2\, \lambda, \end{cases}$$ Airy function λ.

Examples: reflexive modules

- div-curl-grad: $\vec{\nabla}.\,\vec{B} = 0 \;\Leftrightarrow\; \vec{B} = \vec{\nabla} \wedge \vec{A},\; \vec{\nabla} \wedge \vec{A} = \vec{0} \;\Leftrightarrow\; \vec{A} = \vec{\nabla}\, f.$
- First group of Maxwell equations:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}.\,\vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\, V = \vec{0}, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{A} = \vec{\nabla}\, \xi, \\ V = -\dfrac{\partial \xi}{\partial t}. \end{array} \right.$$

- 3D stress tensor: Maxwell, Morera parametrizations. . .
- Linearized Einstein equations (system of PDEs 10×10)?

$\Rightarrow$ OreModules (Chyzak, Robertz, Q.)

Variational problems

- Let us extremize the electromagnetic action

$$\int \left(\frac{1}{2\,\mu_0} \parallel \vec{B} \parallel^2 - \frac{\epsilon_0}{2} \parallel \vec{E} \parallel^2 \right) dx_1\, dx_2\, dx_3\, dt, \quad (1)$$

where $\vec{B}$ and $\vec{E}$ satisfy:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} \,.\, \vec{B} = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\, V - \dfrac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right. \quad (3)$$

- Substituting (3) in (1) and using Lorentz gauge

$$\vec{\nabla} \,.\, \vec{A} + \frac{1}{c^2}\, \frac{\partial V}{\partial t} = 0, \quad c^2 = 1/(\epsilon_0\, \mu_0),$$

$$\Rightarrow \left\{ \begin{array}{l} \dfrac{1}{c^2}\, \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\, \vec{A} = 0, \\ \dfrac{1}{c^2}\, \dfrac{\partial^2 V}{\partial t^2} - \Delta\, V = 0. \end{array} \right. \quad \text{(electromagnetic waves).}$$

Projectiveness, observability and controllability

- Theorem: If $R \in D^{q \times p}$ has full row rank, then the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$ is projective iff:

$$N = D^{1 \times q}/(D^{1 \times p}\, \theta(R)) = 0 \Leftrightarrow \exists\, S \in D^{p \times q} : \; R\, S = I_q.$$

- Let $D = \mathcal{A}(I)\left[\partial; \mathrm{id}, \frac{d}{dt}\right]$ and $R = (\partial\, I_n - A \quad -B) \in D^{n \times (n+m)}$.

$M = D^{1 \times (n+m)}/(D^{1 \times n} R)$ is projective iff $\theta(R)\, \lambda = 0 \Leftrightarrow \lambda = 0$:

$$\left\{ \begin{array}{l} -\partial\, \lambda - A^T\, \lambda = 0, \\ -B^T\, \lambda = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} \partial\, \lambda = -A^T\, \lambda, \\ B^T\, \lambda = 0, \\ B^T\, \partial\, \lambda + \dot{B}^T\, \lambda = (-B^T\, A^T + \dot{B}^T)\, \lambda = 0. \end{array} \right.$$

Hence, M is projective iff, for all $t_0 \in I$, we have:

$$\mathrm{rank}_{\mathbb{R}}(B \mid A\, B - \dot{B} \mid A^2\, B + \ldots \mid A^{n-1}\, B + \ldots \mid \ldots)(t_0) = n.$$

- $D^{1 \times p}/(D^{1 \times q}\, (P(\partial) \;\; -Q(\partial))$ proj. iff $P(\partial)\, X(\partial) - Q(\partial)\, Y(\partial) = I_q$.

Constructive version of the Quillen-Suslin theorem

- Constructive proofs of the Quillen-Suslin theorem exist and one was implemented by Fabiańska in the package QuillenSuslin.

$\Rightarrow$ Computation of bases of free $k[x_1, \ldots, x_n]$-modules.

$\Rightarrow$ Computation of flat outputs of flat systems (Fliess et al).

(Logemann, SCL 87) $$\left\{\begin{array}{l} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\, x_2(t) = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} x_1(t) = \dot{z}(t) - z(t-h) + a\, z(t), \\ x_2(t) = z(t), \\ u(t) = \ddot{z}(t) + \dot{z}(t) - \dot{z}(t-h) - z(t-h) + a\, \dot{z}(t) + a\, z(t). \end{array}\right.$$

- Completion problem: $\left(\begin{array}{c} R \\ T \end{array}\right) \left(\begin{array}{cc} S & Q \end{array}\right) = \left(\begin{array}{cc} I_q & 0 \\ 0 & I_{p-q} \end{array}\right) = I_p.$

A flat time-delay system is equivalent to the system without delay!

Constructive version of Stafford's theorem

- The time-varying linear control system (Sontag)

$$\begin{cases} \dot{x}_1(t) - t\, u_1(t) = 0, \\ \dot{x}_2(t) - u_2(t) = 0, \end{cases}$$

is injectively parametrized by (Stafford, Robertz, Q.)

$$\begin{cases} x_1(t) = t^2\, z_1(t) - t\, \dot{z}_2(t) + z_2(t), \\ x_2(t) = t\,(t+1)\, z_1(t) - (t+1)\, \dot{z}_2(t) + z_2(t), \\ u_1(t) = t\, \dot{z}_1(t) + 2\, z_1(t) - \ddot{z}_2(t), \\ u_2(t) = t\,(t+1)\, \dot{z}_1(t) + (2\,t+1)\, z_1(t) - (t+1)\, \ddot{z}_2(t), \end{cases}$$

and $\{z_1,\, z_2\}$ is a basis of the free left $A_1(\mathbb{Q})$-module M as:

$$\begin{cases} z_1(t) = (t+1)\, u_1(t) - u_2(t), \\ z_2(t) = (t+1)\, x_1(t) - t\, x_2(t). \end{cases}$$

- Idem for $\partial_1\, y_1 + \partial_2\, y_2 + \partial_3\, y_3 + x_3\, y_1 = 0$.

Morphisms and transformations

• We consider $R' \in D^{q' \times p'}$, $\ker_{\mathcal{F}}(R'.)$, $M' = D^{1 \times p'}/(D^{1 \times q'} R')$.

How can we send elements of $\ker_{\mathcal{F}}(R'.)$ to elements of $\ker_{\mathcal{F}}(R.)$?

• Theorem: Any element $f \in \hom_D(M, M')$ is defined by two matrices $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying that:

$$\boxed{R\,P = Q\,R'}$$

• If $f \in \hom_D(M, M')$, then we can define $(R\,P\,\eta' = Q\,R'\,\eta' = 0)$:

$$\begin{array}{rcl} f^\star : \ker_{\mathcal{F}}(R'.) & \longrightarrow & \ker_{\mathcal{F}}(R.), \\ \eta' & \longmapsto & \eta = P\,\eta'. \end{array}$$

• $\hom_D(M, M')$ can be totally (resp., partially) computed if D is a commutative (resp., non-commutative) polynomial ring.

• $\mathrm{end}_D(M) = \hom_D(M, M)$ defines the internal symmetries of M.

Factorization, reduction and decomposition problems

- The knowledge of the ring $\mathrm{end}_D(M) = \hom_D(M, M)$ and of its idempotents allows us to constructively study the problems:

1. $\exists \, R_1 \in D^{r \times p}, \; R_2 \in D^{q \times r} : R = R_2 \, R_1$?
2. $\exists \, W \in \mathrm{GL}_p(D), \; V \in \mathrm{GL}_q(D) : V \, R \, W = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$?
3. $\exists \, W \in \mathrm{GL}_p(D), \; V \in \mathrm{GL}_q(D) : V \, R \, W = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$?

- Basis computations are used for Problems 2 and 3

$\Rightarrow$ OreModules, Jacobson, QuillenSuslin, Stafford.

- Algorithms are implemented in OreMorphisms (Cluzeau, Q.).

Moving tank (Petit, Rouchon, IEEE TAC 02)

$$R = \begin{pmatrix} \partial_1 & -\partial_1\,\partial_2^2 & \alpha\,\partial_1^2\,\partial_2 \\ \partial_1\,\partial_2^2 & -\partial_1 & \alpha\,\partial_1^2\,\partial_2 \end{pmatrix}.$$

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(D), \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

$$\overline{R} = V\,R\,U^{-1} = \begin{pmatrix} \partial_1\,(1-\partial_2^2) & 0 & 0 \\ 0 & \partial_1\,(1+\partial_2^2) & 2\,\alpha\,\partial_1^2\,\partial_2 \end{pmatrix}.$$

Classical systems of PDEs

$$U \begin{pmatrix} \partial_t - k\,\partial_x^2 - a_1 & -b_1 \\ -a_2 & \partial_t - k\,\partial_x^2 - b_2 \end{pmatrix} U^{-1}$$

$$= \begin{pmatrix} \partial_t - k\,\partial_x^2 - \frac{(a_1+b_2)}{2} + \frac{1}{2\,\alpha} & 0 \\ 0 & \partial_t - k\,\partial_x^2 - \frac{(a_1+b_2)}{2} - \frac{1}{2\,\alpha} \end{pmatrix}.$$

$$U = V = \begin{pmatrix} 2\,a_2\,\alpha & (b_2 - a_1)\,\alpha - 1 \\ 2\,a_2\,\alpha & (b_2 - a_1)\,\alpha + 1 \end{pmatrix},$$

$$((a_1 - b_2)^2 + 4\,a_2\,b_1)\,\alpha^2 - 1 = 0.$$

- Wave/Cauchy-Riemann/Dirac/Beltrami eqs, electrical line...

Wind tunnel model (Manitius, IEEE TAC 84)

$$V \begin{pmatrix} \partial_1 + a & -k\,a\,\partial_2 & 0 & 0 \\ 0 & \partial_1 & -1 & 0 \\ 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} U^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \partial_1 + a & -\omega^2\,k\,a\,\partial_2 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & \omega^2 & \partial_1 + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \partial_1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\omega^2} & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Computations obtained by OreMorphisms or Serre (Q.).

String with an interior mass (Fliess et al, COCV 98)

$$(\star)\left\{\begin{array}{l}\phi_1(t)+\psi_1(t)-\phi_2(t)-\psi_2(t)=0,\\ \dot{\phi}_1(t)+\dot{\psi}_1(t)+\eta_1\,\phi_1(t)-\eta_1\,\psi_1(t)-\eta_2\,\phi_2(t)+\eta_2\,\psi_2(t)=0,\\ \phi_1(t-2\,h_1)+\psi_1(t)-u(t-h_1)=0,\\ \phi_2(t)+\psi_2(t-2\,h_2)-v(t-h_2)=0.\end{array}\right.$$

$$V\begin{pmatrix}1 & 1 & -1 & -1 & 0 & 0\\ \partial_1+\eta_1 & \partial_1-\eta_1 & -\eta_2 & \eta_2 & 0 & 0\\ \partial_2^2 & 1 & 0 & 0 & -\partial_2 & 0\\ 0 & 0 & 1 & \partial_3^2 & 0 & -\partial_3\end{pmatrix}U^{-1}$$

$$=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \partial_1+\eta_1+\eta_2 & 2\,\eta_1\,\partial_2 & 2\,\eta_2\,\partial_3\end{pmatrix},$$

$$(\star)\Leftrightarrow \dot{z}_1(t)+(\eta_1+\eta_2)\,z_1(t)+2\,\eta_1\,z_2(t-h_1)+2\,\eta_2\,z_3(t-h_2)=0.$$

String with an interior mass (Fliess et al, COCV 98)

The unimodular matrices U and V are defined by:

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 & \partial_2 & 0 \\ 0 & -1 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\partial_3 \\ 0 & -1 & -1 & 1 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_2 & \partial_2^2 - 1 & 0 \\ 0 & -\partial_3 & -\partial_3 & \partial_3 & 0 & \partial_3^2 - 1 \end{pmatrix},$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_2^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_2^2\,(\partial_1 - \eta_1 + \eta_2) - \partial_1 - \eta_1 & 1 & -\partial_1 + \eta_1 - \eta_2 & 2\,\eta_2 \end{pmatrix}.$$

- The computations were obtained by OreMorphisms or Serre.

Conclusion

• Based on algebraic analysis, constructive homological algebra and Ore algebras, we have developed a general non-commutative polynomial approach to functional linear systems.

• The different results have been implemented in packages:

OreModules, JanetOre, OreMorphisms, Serre, Stafford, QuillenSuslin, Homalg.

This new approach allowed us to:

1. Develop an intrinsic approach (independent of the form).
2. Develop generic algorithms and generic implementations.
3. Constructively study certain classes of flat systems.
4. Extend the concepts of primeness, solve conjectures. . .