A constructive theory of classes and sets

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MAP, ICTP – Trieste, 25-29 August 2008

A problem

In the effective topos $\mathbf{Eff}$, consider the three constant arrows $\{0\} \overset{\{0\}}{\longrightarrow} \mathbb{P}\{0\}$, $\{0\} \overset{0}{\longleftarrow} \mathbb{P}\{0\}$, $\{0\} \overset{\emptyset}{\longrightarrow} \mathbb{P}\{0\}$.

D is *discrete* if one (hence each) evaluation function $D^{\{0\}} \overset{\mathrm{ev}_\emptyset}{\longleftarrow} D^{\mathbb{P}\{0\}}$, $D^{\{0\}} \overset{\mathrm{ev}_0}{\longrightarrow} D^{\mathbb{P}\{0\}}$, $D^{\{0\}} \overset{\mathrm{ev}_{\{0\}}}{\longleftarrow} D^{\mathbb{P}\{0\}}$ is bijective.

The families of discrete objects $(D_i)_{i \in I}$ in $\mathbf{Eff}$ are characterized as having each fibre isomorphic to a quotient of a set of numbers, *i.e.*
$\forall_{i \in I}\ \exists_{Q \in \mathrm{Quot}(\mathbf{N})}\ \exists_{f \in D_i^X}\ f$ is bijective

Though the choice of Q depending on i is basically unique (it is unique up to isomorphism), it is not possible, in general, to find such a Q explicitly for a given i.

The appropriate universe where to study this situation is that of groupoids in $\mathbf{Eff}$.
These form a model of an elementary theory of classes and
it includes a model of extensional type theory.

This is related to work of M. Hofmann and Th. Streicher, *The groupoid interpretation of type theory*, in Oxford Logic Guides 36, 1998.

M. Hyland, E. Robinson, G.R., *The discrete objects in the effective topos*, Pr.L.M.S. 60 (1990)

An Elementary Theory of Classes, I – The Logic

$$P \vdash P$$

if $P \vdash Q$ e $Q \vdash R$, then $P \vdash R$

if $P(x) \vdash Q(x)$, then $P(t) \vdash Q(t)$

These are the logical axioms and rules of the theory, written on a line. A common form to present these and those to follow is $\frac{}{P \vdash P}$ $\frac{P \vdash Q \quad Q \vdash R}{P \vdash R}$ ... $\frac{R \vdash P \quad R \vdash Q}{R \vdash P \wedge Q}$ $\frac{R \vdash P \wedge Q}{R \vdash P}$ ...

$$\bot \vdash R \qquad\qquad R \vdash \top$$

$R \vdash P \wedge Q$ if and only if $R \vdash P$ e $R \vdash Q$

$P \vee Q \vdash R$ if and only if $P \vdash R$ e $Q \vdash R$

$R \vdash P \Rightarrow Q$ if and only if $R \wedge P \vdash Q$

$R \vdash \forall_{x \in A} P(x)$ if and only if $x \in A \wedge R \vdash P(x)$

$\exists_{x \in A} P(x) \vdash R$ if and only if $x \in A \wedge P(x) \vdash R$

$[x \in A \wedge R(x)] \wedge x = y \vdash Q(x, y)$ if and only if $x \in A \wedge R(x) \vdash Q(x, x)$

$\neg P \vdash Q$ if and only if $\neg Q \vdash P$

$$\neg P \overset{\text{def}}{\Leftrightarrow} P \Rightarrow \bot$$

F. W. Lawvere, *Adjointness in foundations*, Dialectica 23 (1969)

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This is the only rule that must be removed to axiomatize the intuitionistic version of the theory.

[$\neg P \vdash Q$ if and only if $\neg Q \vdash P$]

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An Elementary Theory of Classes, II – The Classes

Basic constructions pair: $\langle a, b\rangle$ projections: x_1 x_2 abstraction: $\{x \mid P(x)\}$

$$\{x \mid P(x)\} \times \{x \mid Q(x)\} \stackrel{\text{def}}{=} \{z \mid z = \langle z_1, z_2\rangle \wedge (P(z_1) \wedge Q(z_2))\}$$

$$\mathbb{U} \stackrel{\text{def}}{=} \{x \mid \top\}$$

$$\{x \mid P(x)\} \text{ is a class} \stackrel{\text{def}}{\Leftrightarrow} \forall_{y\in\mathbb{U}}\ [y \in \{x \mid P(x)\} \Leftrightarrow P(y)]$$

A is a class $\Leftrightarrow A = \{x \mid x \in A\}$ thanks to the second axiom for equality.

Axioms for equality $\langle x, y\rangle_1 = x$ $\langle x, y\rangle_2 = y$

$$\{x \mid P(x)\} = \{x \mid Q(x)\} \Leftrightarrow \forall_{x\in\mathbb{U}}\ [P(x) \Leftrightarrow Q(x)]$$

Axioms for classes $\mathbb{U}$ is a class

if A, B are classes, then $A \times B$ is a class

if A is a class, then $\{x \in A \mid t = s\}$ is a class

if A is a class, $\forall_{a\in A}\ B_a$ is a class, then $\{x \in A \mid t \in B_x\}$ is a class

if A is a class, $\forall_{a\in A}\ B_a$ is a class, then $\bigcup_{a\in A} B_a$ is a class

A. Joyal and I. Moerdijk, *Algebraic Set Theory*, LMS 220, Cambridge University Press, 1995

An Elementary Theory of Classes, III – The Sets

Basic notion X is a set

Axioms for sets
- if X is a set, then X is a class
- if A is a class, X is a set, $f\colon A \simeq X$, then A is a set
- if X is a set, Y is a set, then $X \times Y$ is a set
- if A is a class, X is a set, $A \subseteq X$, then A is a set
- if I is a set, $\forall_{i \in I}\ X_i$ is a set, then $\bigcup_{i \in I} X_i$ is a set

$$\mathbb{P}(A) \stackrel{\text{def}}{=} \{Z \mid Z \text{ is a set} \wedge Z \subseteq A\}$$

The notion of inclusion is defined as usual, as are those of function and bijection.

Powerset Axioms
- if A is a class, then $\mathbb{P}(A)$ is a class
- if X is a set, then $\mathbb{P}(X)$ is a set

Axiom of infinity There are a set $\mathbb{N}$, $0 \in \mathbb{N}$, $s\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that
- $\forall_{n \in \mathbb{N}}\ 0 \neq s(n)$
- $\forall_{n,n' \in \mathbb{N}}\ [s(n) = s(n') \Rightarrow n = n']$
- $\forall_{X \in \mathbb{P}(\mathbb{N})}\ [[0 \in X \wedge \forall_{x \in X}\ s(x) \in X] \Rightarrow X = \mathbb{N}]$

A. Simpson, *Elementary axioms for categories of classes*, in Procs. LICS XIV, 1999

Realizing the theory

Fix a class U such that, for $a, b \in U$, also $\langle a, b\rangle \in U$, and
a relation $\underline{\mathtt{r}}_U \subseteq \mathbb{N} \times U$ such that, for $n\ \underline{\mathtt{r}}_U\ a, m\ \underline{\mathtt{r}}_U\ b$, it is $(n, m)\ \underline{\mathtt{r}}_U\ \langle a, b\rangle$.

This is notation for a chosen recursive encoding of pairs of numbers.

A *realizable* assertion is a relation $P \subseteq \mathbb{N} \times U$.
A *realizable* class is a subclass of U.

For P, Q realizable assertions, say that

$n\ \underline{\mathtt{r}}_U\ (P \vdash Q)$ if

for all $x \in U$, for all $k\ \underline{\mathtt{r}}_U\ x$, for all $\ell\ P\ x$
the Turing machine M_n, encoded by n, is defined on (k, ℓ)
and $M_n(k, \ell)\ Q\ x$

$P \vdash Q$ is *realized* if there is $n\ \underline{\mathtt{r}}_U\ (P \vdash Q)$.

S. Kleene, *On the interpretation of intuitionistic number theory*, J.Symb.Logic 10 (1945)
J.M.E. Hyland, *The effective topos*, in Procs. L.E.J. Brouwer Centenary Symposium, 1982
J. van Oosten, *Realizability*, Oxford University Press, 2008

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and $M_n(k, \ell) \; Q \; x$

$P \vdash Q$ is *realized* if there is $n \;\underline{\mathtt{r}}_U\; (P \vdash Q)$.

The realization of connectives and quantifiers is somehow forced by the need to verify the logical axioms.

$$n \; (P \wedge Q) \; x \overset{\text{def}}{\Leftrightarrow} \begin{cases} n = (p, q) \\ p \; P \; x \\ q \; Q \; x \end{cases} \qquad n \; (P \vee Q) \; x \overset{\text{def}}{\Leftrightarrow} \begin{cases} n = (p, q) \\ p = 0 \Rightarrow q \; P \; x \\ p = 1 \Rightarrow q \; Q \; x \end{cases}$$

$$n \; (P \Rightarrow Q) \; x \overset{\text{def}}{\Leftrightarrow} \forall_{k \underline{\mathtt{r}}_U x} \forall_{\ell P x} \; M_n(k, \ell) \; Q \; x$$

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Realizing the Theory of Classes and Sets

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~~$\neg P \vdash Q$ if and only if $\neg Q \vdash P$~~

This is the only rule that fails to be realized.
But, after all, realizability is an excellent semantics for intuitionistic theories.

Axioms for equality $\langle x, y\rangle_1 = x \qquad \langle x, y\rangle_2 = y$

$$\{x \mid P(x)\} = \{x \mid Q(x)\} \Leftrightarrow \forall_{x\in\mathbb{U}}\ [P(x) \Leftrightarrow Q(x)]$$

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~~if I is a set, $\forall_{i\in I}\ X_i$ is a set, then $\bigcup_{i\in I} X_i$ is a set~~

Powerset Axioms

if A is a class, then $\mathbb{P}(A)$ is a class

if X is a set, then $\mathbb{P}(X)$ is a set

But these two axioms fail too. And, in the last, the notion of subset is no longer clear.

Axiom of infinity

There are a set $\mathbf{N}$, $\quad 0 \in \mathbf{N}$, $\quad s\colon \mathbf{N} \to \mathbf{N}$ such that

- $\forall_{n\in\mathbf{N}}\ 0 \neq s(n)$
- $\forall_{n,n'\in\mathbf{N}}\ [s(n) = s(n') \Rightarrow n = n']$

× $\forall_{X\in\mathbb{P}(\mathbf{N})}\ [[0 \in X \wedge \forall_{x\in X}\ s(x) \in X] \Rightarrow X = \mathbf{N}]$

Correcting the realization of the theory

Take the smallest class R of sets $a \in \mathbb{P}(\mathbb{U} \times \mathbb{N})$ such that

- $\forall_{b,c \in R} \, \forall_{m \in \mathbb{N}} \, [m\underline{\mathtt{r}}(b \Bumpeq c) \Rightarrow [b \in \mathsf{dom}(a) \Leftrightarrow c \in \mathsf{dom}(a)]]$
- $\exists_{t \in \mathbb{N}} \, \forall_{b,c \in R} \, t\underline{\mathtt{r}}[b \Bumpeq c \Rightarrow [b \, \varepsilon \, a \Leftrightarrow c \, \varepsilon \, a]]$

Different notations for equality and membership under realizability are employed.

where

- $m\underline{\mathtt{r}}(b \Bumpeq c) \overset{\mathsf{def}}{\Leftrightarrow} \forall_{x \in R} \, m\underline{\mathtt{r}}[x \, \varepsilon \, b \Leftrightarrow x \, \varepsilon \, c]$
- $(p,q)\underline{\mathtt{r}}(d \, \varepsilon \, b) \overset{\mathsf{def}}{\Leftrightarrow} [\langle d, p \rangle \in b \wedge \forall_{x \in R} \, q\underline{\mathtt{r}}[d \Bumpeq x \Rightarrow [d \, \varepsilon \, b \Leftrightarrow x \, \varepsilon \, b]]$

Otherwise, giving up the universe class, define classes as pairs $A = (|A|, [\![=_A]\!])$ where

- $|A|$ is a set
- $[\![=_A]\!] \subseteq |A| \times |A| \times \mathbb{N}$ for which simmetry and transitivity are realized.