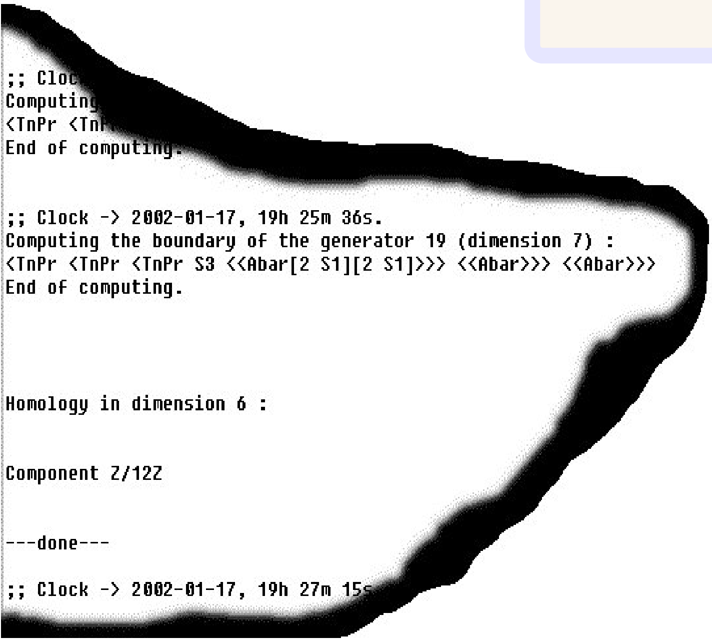
Effective Constructive

Algebraic Topology



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Map Ictp Conference, Trieste, August 25-29, 2008

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Three solutions for Constructive Algebraic Topology:

1. **Rolf Schön (Inductive methods).**
2. **Effective Homology.**
3. **Operadic Algebraic Topology.**

Only the second one so far led to concrete computer programs.

Plan of the talk: 1. Computer illustration around CW-complexes.

2. Constructive statement of the homological problem.

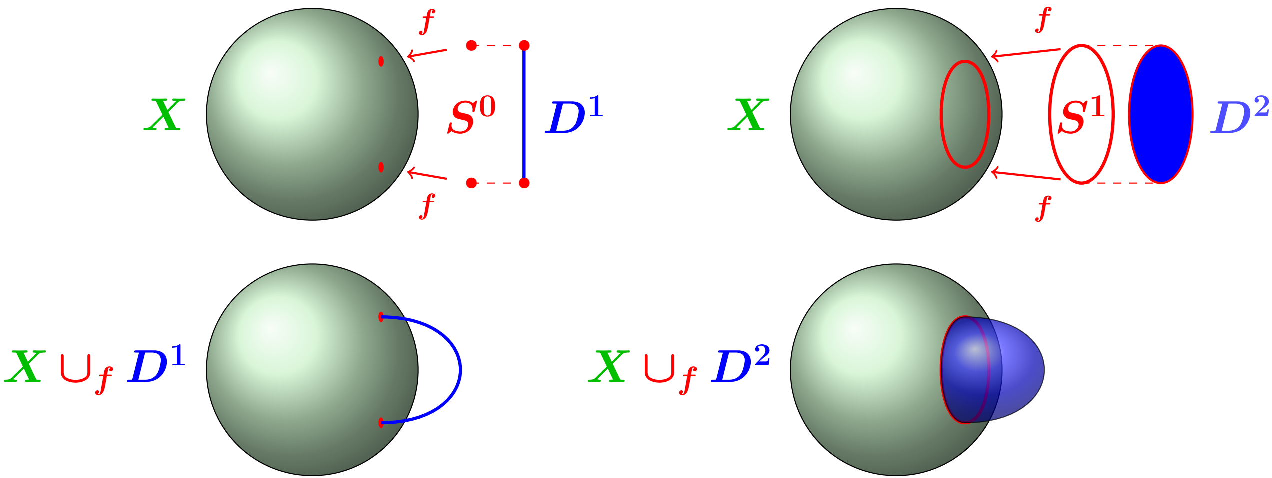
3. Other computer illustrations.

Attaching a cell D^n to a topological space X along the boundary S^{n-1}:

X = Topological space.

$f : S^{n-1} \to X$ = continuous map.

$\Rightarrow X \cup_f D^n := (X \coprod D^n)/(X \ni f(x) \sim x \in S^{n-1})$.



Notion of CW-Complex X:

$$X = \varinjlim \{X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_n \subset \cdots\}_{n \in \mathbb{N}}$$

with X_0 = discrete space and

the n-skeleton X_n is obtained

from the $(n-1)$-skeleton X_{n-1}

by attaching n-disks $D_1^n, D_2^n, \cdots$ to X_{n-1}

according to attaching maps $f_1^n, f_2^n, \cdots$

Every reasonable space can be presented

up to homotopy equivalence

as a CW-complex of finite type.

Example 1. Presentation of $X = P^2\mathbb{R}$ as a CW-complex.

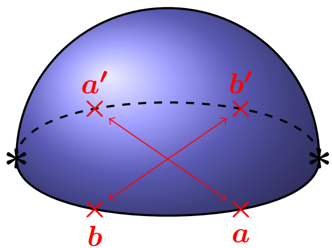
$$X_0 = *$$

$$D^1 \supset S^0 \xrightarrow{f^1} *$$

$$\Rightarrow X_1 = X_0 \cup_{f^1} D^1 = X_1 = S^1 = \{Z \in \mathbb{C} \ \underline{\text{st}} \ |z| = 1\}$$

$$D^2 \supset S^1 \xrightarrow{f^2} S^1 : z \mapsto z^2$$

$$\Rightarrow X = X_2 = X_1 \cup_{f^2} D^2 = P^2\mathbb{R}$$

$P^2\mathbb{R} =$ 

$a \sim a'$

$b \sim b'$

Example 2. More generally:

Presentation of $X = P^\infty\mathbb{R}$ as a CW-complex.

1. $X_0 = P^0\mathbb{R} = S^0/\!\sim\, = *$.

2. Let us assume $X_n = P^n\mathbb{R}$ constructed.

3. $D^{n+1} \supset S^n \xrightarrow{f^{n+1}} P^n\mathbb{R}$

   with f^{n+1} = the canonical projection.

4. $\Rightarrow X_{n+1} = D^{n+1} \cup_{f^{n+1}} X_n = P^{n+1}\mathbb{R}$.

$(++\, n)$; goto 2.

5. $X = \lim_{\rightarrow} X_n = P^\infty\mathbb{R}$.

Example 3. Simplicial complexes and simplicial sets.

X = **simplicial set.**

Definition: **The n-skeleton X_n of X is obtained from X by keeping the non-degenerate simplices of dimension $\leq n$ (and their degeneracies), throwing away the non-degenerate simplices of dimension $> n$ (and their degeneracies).**

$|X_n|$ **obtained from** $|X_{n-1}|$

by attaching n-simplices = n-disks.

$\Rightarrow X$ = **CW-complex with** $|X| = \lim_{\rightarrow} |X_n|$.

Simplicial version of $P^\infty\mathbb{R}$:

$$P^\infty\mathbb{R} = X = K(\mathbb{Z}_2, 1)$$

$$\Rightarrow X_n^{ND} = \{\sigma_n\}$$

$$\begin{aligned} \partial_i \sigma_n &= \sigma_{n-1} \text{ if } i = 0, n; \\ &= \eta_{i-1}\sigma_{n-2} \text{ if } 0 < i < n. \end{aligned}$$

$$\Rightarrow C_*X = \{\cdots \longleftarrow \overset{0}{\mathbb{Z}} \overset{0}{\longleftarrow} \overset{1}{\mathbb{Z}} \overset{\times 2}{\longleftarrow} \overset{2}{\mathbb{Z}} \overset{0}{\longleftarrow} \overset{3}{\mathbb{Z}} \overset{\times 2}{\longleftarrow} \overset{4}{\mathbb{Z}} \overset{0}{\longleftarrow} \overset{5}{\mathbb{Z}} \overset{\times 2}{\longleftarrow} \cdots\}$$

$$\begin{aligned} \Rightarrow H_i(P^\infty\mathbb{R}) &= \mathbb{Z} \quad \text{if } i = 0; \\ & \mathbb{Z}_2 \quad \text{if } i > 0 \text{ odd}; \\ & 0 \quad \text{if } i > 0 \text{ even}. \end{aligned}$$

The same for $P^\infty\mathbb{C}$?

Topological version ? Easy.

$P^\infty\mathbb{C} = X = \lim_{\rightarrow} X_{2n}$ where:

$$X_{2n} = X_{2n-2} \cup_{f^{2n}} D^{2n}$$

with: $D^{2n} \supset S^{2n-1} \longrightarrow P^{n-1}\mathbb{C}$ the canonical projection.

Simplicial version?

Much harder!

Easy up to homotopy.

Easiest solution = $\boldsymbol{K(\mathbb{Z},2)}$.

Justification = two principal fibrations:

$$\boldsymbol{S^1 \hookrightarrow S^\infty \longrightarrow P^\infty\mathbb{C}}$$

$$\boldsymbol{K(\mathbb{Z},1) \hookrightarrow E(\mathbb{Z},1) \longrightarrow K(\mathbb{Z},2)}$$

$+\ (\boldsymbol{K(\mathbb{Z},1) \sim S^1})+(\boldsymbol{S^\infty}$ **contractible**$)\ +(\boldsymbol{E(\mathbb{Z},1)}$ **contractible**$)$

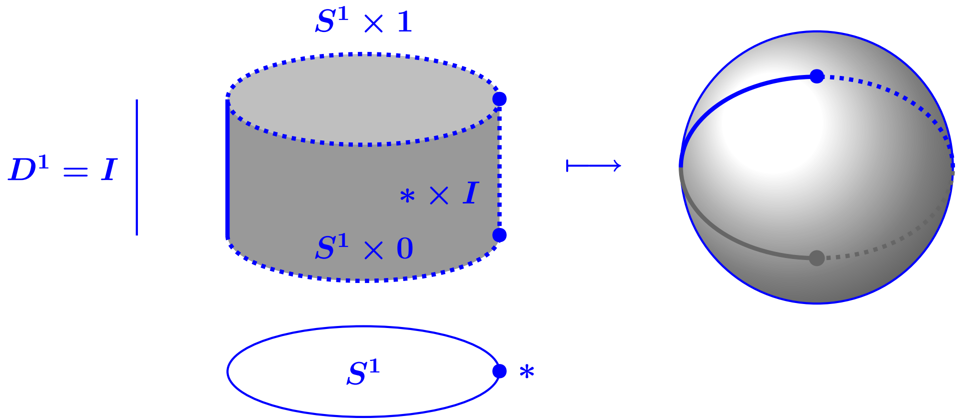
$$\Rightarrow \boldsymbol{P^\infty\mathbb{C} \sim K(\mathbb{Z},2)}$$

Remark: $\boldsymbol{K(\mathbb{Z},2)}$ **not of finite type!**

Simplicial model of finite type for $\boldsymbol{P^2\mathbb{C}}$**??**

Cellular homology.

$S^n = S^1 \times D^{n-1}/\sim$ with $(z,x) \sim (z',x')$ if $x = x' \in \partial D^{n-1}$.



Canonical self-map of degree k for S^n:

$$\alpha_k : S^n \to S^n : (z,x) \mapsto (z^n, x).$$

<u>Theorem</u> (Hopf): $\mathcal{C}(S^n, S^n)/\sim \; \cong \mathbb{Z}$.

CW-complex:

$$X = \varinjlim X_n = \{(D_i^n, f_i^n : S^{n-1} \longrightarrow X_{n-1})_{1 \le i \le m_n}\}_{n \in \mathbb{N}}$$

Associated cellular chain complex:

$$\cdots \longleftarrow \mathbb{Z}^{(m_{n-1})} \xleftarrow{d_n} \mathbb{Z}^{(m_n)} \longleftarrow \cdots$$

Coefficient $\alpha_{1,1}$ of d_n in column 1 and row 1

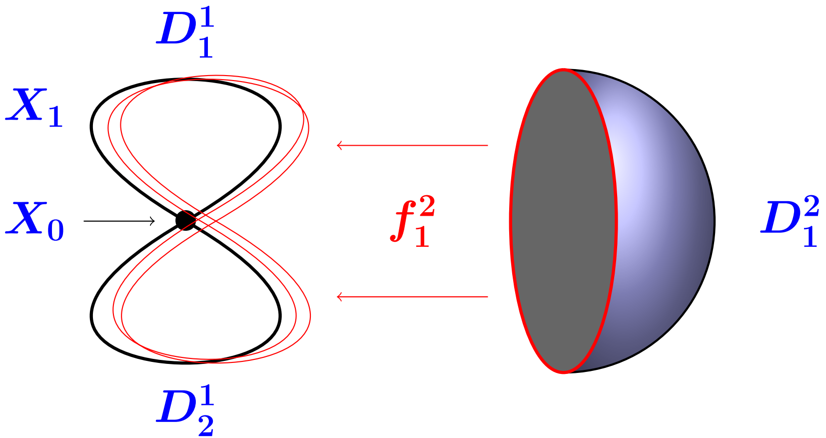
obtained from $g_{1,1}^n$:

$$f_1^n : S^{n-1} \longrightarrow X^{n-1}$$

$$\Rightarrow g_{1,1}^n : S^{n-1} \longrightarrow Y_1^{n-1} = X^{n-1}/[X^{n-2} \cup (\cup_{i \neq 1} D_i^{n-1})] = S^{n-1}$$

$$\Rightarrow \alpha_{1,1} = \deg(g_{1,1}^n).$$

Example: $\boldsymbol{X} =$



Cellular complex $= \{0 \longleftarrow \mathbb{Z} \xleftarrow{d_1} \mathbb{Z}^2 \xleftarrow{d_2} \mathbb{Z} \longleftarrow 0\}$

with $d_1 = [0\ 0]$ **and** $d_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow H_* = \{\mathbb{Z}, \mathbb{Z}_2 + \mathbb{Z}, 0\}$

Theorem (Adams, 1956): Let X be a 1-reduced CW-complex (one vertex, no 1-cell).

Then ∃ a CW-model for the loop space ΩX, where every sequence $(\sigma_1, \ldots, \sigma_k)$ of cells of X of respective dimensions $(d_1, \ldots, d_k)$ generate a cell of dimension $(d_1 + \cdots + d_k - k)$ in the CW-model of ΩX.

Examples:

$$S^3 = (*, 0, 0, 1) \;\Rightarrow\; \Omega S^3 = (*, 0, 1, 0, 1, 0, 1, \ldots).$$

$$P^2\mathbb{C} = (*, 0, 1, 0, 1) \Rightarrow$$

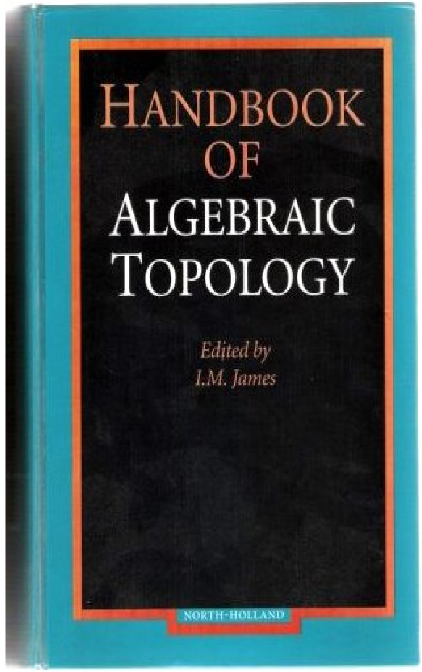
$$\Omega P^2\mathbb{C} = (*, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \ldots).$$

Typical example

extracted from

the encyclopedy:

(Ioan James editor).



Chapter 13

Stable Homotopy and Iterated Loop Spaces

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Stable Homotopy and Iterated Loop Spaces

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HANDBOOK OF ALGEBRAIC TOPOLOGY
Edited by I.M. James

6. The structure of second loop spaces

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of X, which is homotopy equivalent to ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume X has no i-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adams' solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

Translation: **No known algorithm using these methods computes $H_*(\Omega^n X)$ for $n \geq 3$ except when X is an n-suspension $X = \Sigma^n Y$.**

Typical example: $H_*(\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})) =$???

Adams: **There exists a finite-type CW-complex with the homotopy type of** $\Omega^3(P^\infty\mathbb{R}/P^3\mathbb{R})$.

Dimension	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
Cell-#	1	1	2	5	13	33	84	214	545	1388	3535	$\cdots$

But what about the homological boundary matrices ???

Kenzo computing $d_5 : [C_5(\Omega^3) = \mathbb{Z}^{33}] \rightarrow [C_4(\Omega^3) = \mathbb{Z}^{13}]$:

=========== MATRIX 13 lines + 33 columns =====

L1=[C1=-2]

L2=[C1=-1]

L3=[C1=-4][C2=1][C3=-1][C4=-2]

L4=[C2=1][C3=-1][C6=2]

L5=[C1=6][C4=1][C6=1]

L6=[C1=4][C4=4][C6=4][C7=3]

L7=[C1=4][C12=-2][C14=2]

L8=[C1=6][C4=1][C6=1]

L9=[C1=4][C4=4][C6=4][C7=3]

L10=[C8=4][C10=1][C11=-1][C14=-4][C15=-2][C20=-2]

L11=[C1=4][C8=4][C10=1][C11=-1][C16=-4][C18=-1][C19=1][C23=-2]

L12=[C12=4][C13=2][C16=-4][C18=-1][C19=1][C27=-2]

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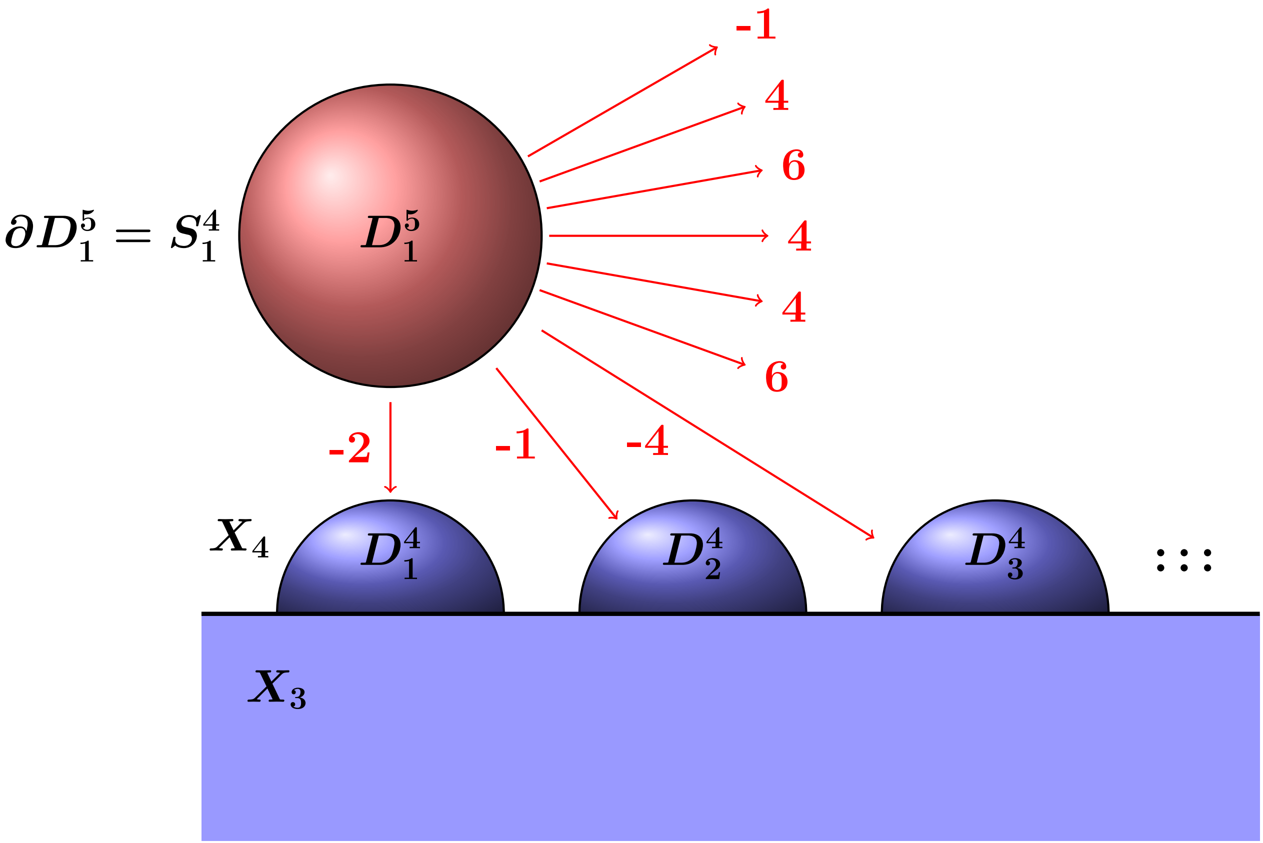
Meaning:



Analysis of the problem:

"Standard" homological algebra is not constructive.

Typical statement:

The sequence $A \xleftarrow{\alpha} B \xleftarrow{\beta} C$ is exact.

Common translation:

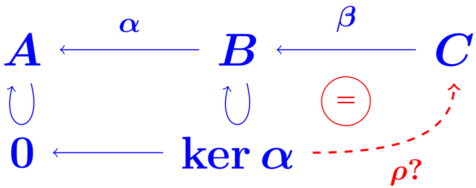
$$(\forall b \in B) \quad [(\alpha(b) = 0) \Rightarrow (\boxed{\exists c \in C} \ \underline{\text{st}} \ b = \beta(c))]$$

with $\boxed{\exists c \in C}$ most often non-constructive.

Constructive exactness:

$$A \xleftarrow{\alpha} B \xleftarrow{\beta} C \text{ constructively exact}$$

if an algorithm $\rho : \ker \alpha \longrightarrow C$ is given satisfying:



$\Rightarrow$ **Organizational algebraic problems:**

where ρ cannot be a group homomorphism.

Definition: **A (homological) reduction is a diagram:**

$$\rho\colon \quad \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ **and** C_* **= chain complexes.**
2. f **and** g **= chain complex morphisms.**
3. h **= homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}} h + h d_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$**.**
5. $fh = 0$, $hg = 0$ **and** $hh = 0$**.**

Let ρ: $\boxed{h \circlearrowleft \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$ be a reduction.

Frequently:

1. **$\widehat{C}_*$ is a locally effective chain complex:**

   its homology groups are unreachable.

2. **C_* is an effective chain complex:**

   its homology groups are computable.

3. **The reduction ρ is an entire description of**

   the homological nature of $\widehat{C}_*$.

4. **Any homological problem in $\widehat{C}_*$ is solvable**

   thanks to the information provided by ρ.

ρ: $\boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$

1. What is $H_n(\widehat{C}_*)$? Solution: Compute $H_n(C_*)$.

2. Let $x \in \widehat{C}_n$. Is x a cycle? Solution: Compute $d_{\widehat{C}_*}(x)$.

3. Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. Let $x, x' \in \widehat{C}_n$ be homologous cycles.

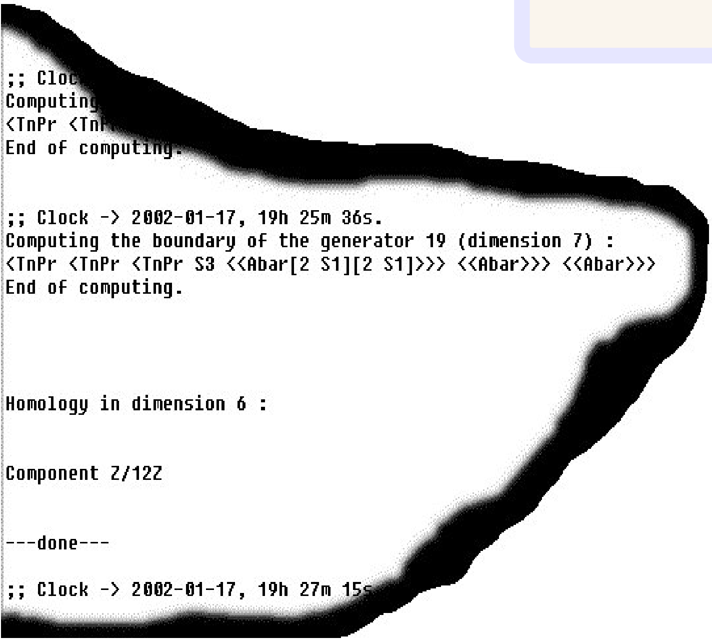
   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.

The END



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