Solving Nonlinear Polynomial Systems via Symbolic-Numeric Elimination Method

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Joint work with Greg Reid and Xiaoli Wu

Zero Dimensional Polynomial System Solving

Consider a polynomial system $F \in \mathbb{C}[x_1, \ldots, x_s]$ of degree d,

$$F : \begin{cases} f_1(x_1, \ldots, x_s) & = & 0, \\ f_2(x_1, \ldots, x_s) & = & 0, \\ & \vdots & \\ f_t(x_1, \ldots, x_s) & = & 0. \end{cases}$$

We are going to find all the common solutions of F.

- Symbolic Algorithms: Gröbner bases, Characteristic sets, Involutive bases, H-bases, Border bases...

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- Numeric Algorithms: Newton's method, Homotopy continuation, Optimization methods...
- Symbolic-Numeric Hybrid Approaches: Gröbner bases, Involutive system, Border bases, Resultant...

Matrix Eigenproblems

F can be written in terms of its coefficient matrix $M_d^{(0)}$ as

$$M_d^{(0)} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1}x_2 \\ \vdots \\ x_s^2 \\ x_1 \\ \vdots \\ x_s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

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Remark: $[\xi_1, \xi_2, \ldots, \xi_s]$ is a solutions of the polynomial system F $\Longleftrightarrow [\xi_1^d, \xi_1^{d-1}\xi_2, \ldots, \xi_s^2, \xi_1, \ldots, \xi_s, 1]^T$ is a null vector of the coefficient matrix $M_d^{(0)}$.

Translates F into a System of PDEs R

The bijection

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \le i \le s,$$

ϕ maps the system F to an equivalent system of linear PDES R:

$$M_d^{(0)} \cdot \begin{pmatrix} \frac{\partial^d u}{\partial x_1^d} \\ \frac{\partial^d u}{\partial x_1^{d-1} \partial x_2} \\ \vdots \\ \frac{\partial^2 u}{\partial x_s^2} \\ \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_s} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Why Differential Equations? [Sturmfels'02]

- We do not lose any information by doing so.

  Remark: A vector $\xi = [\xi_1, \xi_2, \ldots, \xi_s] \in \mathbb{C}^s$ is a solution to polynomial system F if and only if the exponential function $\exp(\xi \cdot x) = \exp(\xi_1 x_1 + \cdots + \xi_s x_s)$ is a solution of the differential equations R.

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- PDE formulation reveals more information than the polynomial formulation.

Example 1 [Sturmfels'02]

Consider the system of three polynomial equations

$$x^3 = yz, \quad y^3 = xz, \quad z^3 = xy.$$

We translate them to the three differential equations

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial y \partial z}, \quad \frac{\partial^3 u}{\partial y^3} = \frac{\partial^2 u}{\partial x \partial z}, \quad \frac{\partial^3 u}{\partial z^3} = \frac{\partial^2 u}{\partial x \partial y}.$$

Example 1 (continued)

A solution basis for the PDEs is given by

$$\exp(x+y+z), \exp(x-y-z), \exp(y-x-z), \exp(z-x-y),$$
$$\exp(x+\boldsymbol{i}y-\boldsymbol{i}z), \exp(x-\boldsymbol{i}y+\boldsymbol{i}z), \exp(y+\boldsymbol{i}x-\boldsymbol{i}z), \exp(y-\boldsymbol{i}x+\boldsymbol{i}z),$$
$$\exp(z+\boldsymbol{i}x-\boldsymbol{i}y), \exp(z-\boldsymbol{i}x+\boldsymbol{i}y), \exp(-x+\boldsymbol{i}y+\boldsymbol{i}z), \exp(-x-\boldsymbol{i}y-\boldsymbol{i}z),$$
$$\exp(-y+\boldsymbol{i}x+\boldsymbol{i}z), \exp(-y-\boldsymbol{i}x-\boldsymbol{i}z), \exp(-z+\boldsymbol{i}y+\boldsymbol{i}x), \exp(-z-\boldsymbol{i}y-\boldsymbol{i}x),$$
$$1, x, y, z, z^2, y^2, x^2, x^3+6yz, y^3+6xz, z^3+6xy, x^4+y^4+z^4+24xyz$$

- The system has 17 distinct complex zeros, 5 are real.

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- The system has 17 distinct complex zeros, 5 are real.
- The multiplicity of the origin $(0,0,0)$ is eleven.
- Polynomial solutions gotten from $x^4+y^4+z^4+24xyz$ by taking successive derivatives describe the multiplicity structure of the polynomial system at $(0,0,0)$.

Symbolic-Numeric Completion of PDEs

We study the linear mapping:

$$R : v \mapsto M_d^{(0)} v$$

here $v = [\underset{d}{u}, \underset{d-1}{u}, \ldots, \underset{1}{u}, u]^T$, and $\underset{j}{u}$ denotes the formal jet coordinates corresponding to derivatives of order exactly j.

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Jet space approaches study the jet variety

$$V(R) = \{(\underset{d}{u}, \underset{d-1}{u}, \ldots, \underset{1}{u}, u) \in J^d : R(\underset{d}{u}, \underset{d-1}{u}, \ldots, \underset{1}{u}, u) = 0\}$$

where $J^d \approx C^{s_d}$ is a jet space of order d and $s_d = \begin{pmatrix} s+d \\ d \end{pmatrix}$.

Symbolic Prolongation

A single prolongation of a system R is defined as:

$$DR := \{w \in J^{d+1} : R(w) = 0, \quad D_{x_1}R(w) = 0, \ldots, D_{x_s}R(w) = 0\}$$

The prolonged system DR has order $d+1$,

$$M_d^{(1)} \cdot \begin{pmatrix} \underset{d+1}{u} \\ \underset{d}{u} \\ \vdots \\ \underset{1}{u} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Remark: $M_d^{(1)}$ is the coeff. matrix of the prolonged system

$$F^{(1)} = F \cup x_1 F \cup \cdots \cup x_s F$$

Geometric Projection

A single geometric projection is defined as

$$\pi(R) := \{(\underset{d-1}{u}, \ldots, \underset{1}{u}, u) \in J^{d-1} : R(\underset{d}{u}, \underset{d-1}{u}, \ldots, \underset{1}{u}, u) = 0\}$$

Remark: For polynomial system, the projection is equivalent to eliminating the monomials of the highest degree d.

- Symbolic elimination method using Gröbner basis algorithms or Ritt-Wu's characteristic algorithms.

  Remark: variables have to be well ordered.

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$$\pi(R) := \{(u_{d-1}, \ldots, u_1, u) \in J^{d-1} : R(u_d, u_{d-1}, \ldots, u_1, u) = 0\}$$

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- Symbolic elimination method using Gröbner basis algorithms or Ritt-Wu's characteristic algorithms.

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- Numerical projection via singular value decomposition.

  Remark: variables are partially ordered by total degrees.

Numeric Projection via SVD

For a chosen tolerance τ:

- Compute SVD of $M_d^{(0)}$, obtain a basis for the null space of R and $\dim R = \dim \text{Nullspace}(M_d^{(0)})$;

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- Estimate $\dim \pi(R)$ by applying SVD to the spanning set of $\pi(R)$.

Numeric Projection via SVD

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- Estimate $\dim \pi(R)$ by applying SVD to the spanning set of $\pi(R)$.
- Proceeding in the same way, we can estimate $\dim \pi^\ell(D^k R)$.

Symbol Involutive

The Symbol matrix of PDEs is Jacobian matrix of the system w.r.t. its highest order jet coordinates,

$$\dim\left(\mathrm{Symbol}\,\pi^{\ell}(D^kR)\right) = \dim\,\pi^{\ell}(D^kR) - \dim\pi^{\ell+1}(D^kR)$$

Remark: In case of polynomials, the Symbol matrix is the submatrix of the coefficient matrix of the system corresponding to highest degree monomials.

Theorem 1. *[Seiler 2002] For finite type PDEs (i.e., zero dimensional polynomial systems), the Symbol of* $\pi^{\ell}(D^kR)$ *is involutive if*

$$\dim\pi^{\ell}(D^kR) = \dim\pi^{\ell+1}(D^kR)$$

Involutive System

The system $R=0$ is said to be involutive at prolonged order k and projected order l, if $\pi^{\ell}(D^k(R))$ satisfies:

$$\dim \pi^{\ell}(D^k R) = \dim \pi^{\ell+1}(D^{k+1} R)$$

and the Symbol of $\pi^{\ell}(D^k R)$ is involutive.

Theorem 2. *[Cartan-Kuranishi 1957] Two integers $k,l \geq 0$ exist for every regular differential equation R such that $\pi^{\ell}(D^k R)$ is involutive.*

Zero Dimensional Polynomial System

Theorem 3. *[Zhi and Reid 2004] If the polynomial system has only finite number of solutions, then* $\pi^\ell(D^k R)$ *is involutive if and only if it satisfies:*

$$\dim \pi^\ell(D^k R) = \dim \pi^{\ell+1}(D^{k+1} R) \quad (\textit{projected elimination test})$$

$$\dim \pi^\ell(D^k R) = \dim \pi^{\ell+1}(D^k R) \quad (\textit{symbol involutive test})$$

and by the bijection ϕ*, we have*

$$\mathrm{numsols}(F) = \dim \pi^\ell(D^k R)$$

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- The number of solutions of polynomial system F is $\dim \pi^{\ell}(D^k R)$.

- Apply eigenvalue method to the null space of $\pi^{\ell}(D^k R)$, $\pi^{\ell+1}(D^k R)$ to obtain the solutions of F.

Example 1 (continue)

Table 1: $\dim \pi^{\ell}(D^k R)$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$\ell=0$	17	23	26	27	27	27
$\ell=1$	10	17	23	26	27	27
$\ell=2$	4	10	17	23	26	27
$\ell=3$	1	4	10	17	23	26
$\ell=4$		1	4	10	17	23
$\ell=5$			1	4	10	17
$\ell=6$				1	4	10
$\ell=7$					1	4
$\ell=8$						1

Example 1 (continued): Eigenvalue Method

We obtain 16 simple solutions:

$$(1,1,1),(1,-1,-1),(1,-1,-1),(-1,-1,1),$$
$$(1,i,-i),(1,-i,i),(i,1,-i),(-i,1,i),$$
$$(i,-i,1),(-i,i,1),(-1,i,i),(-1,-i,-i),$$
$$(i,-1,i),(-i,-1,-i),(i,i,-1),(-i,-i,-1)$$

and one multiple root

$$(0,0,0)$$

with multiplicity 11.

Computing Multiplicity Structure [Wu and Zhi'08]

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- Refine an approximate singular solution

Theorem 4. *[Van Der Waerden 1970] Suppose the polynomial ideal* $I = (f_1, \ldots, f_t)$ *has an isolated primary component* Q *whose associated prime* P *is maximal, and* ρ *is the index of* Q*, i.e., the minimal nonnegative integer s.t.* $\sqrt{Q}^{\rho} \subset Q$*.*

- *If* $\sigma \leq \rho$*, then*

$$\dim(\mathbb{C}[\mathbf{x}]/(I, P^{\sigma-1})) < \dim(\mathbb{C}[\mathbf{x}]/(I, P^{\sigma})).$$

- *If* $\sigma > \rho$*, then*

$$Q = (I, P^{\rho}) = (I, P^{\sigma}).$$

Corollary 5. *The index is less than or equal to the multiplicity* μ*:*

$$\rho \leq \mu = \dim(\mathbb{C}[\mathbf{x}]/Q).$$

Compute Primary Component I

- Form the prime ideal $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$.

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- Compute $d_k = \dim(\mathbb{C}[\mathbf{x}]/(I, P^k))$ by SNEPSolver for a given tolerance τ until $d_k = d_{k-1}$, set

$$\rho = k - 1,\ \mu = d_\rho,\ Q = (I, P^\rho).$$

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$$\rho = k - 1,\ \mu = d_\rho,\ Q = (I, P^\rho).$$

- Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ of $\mathbb{C}[\mathbf{x}]/Q$ to obtain the primary component.

Example 2 [Ojika 1987]

$$I = (f_1 = x_1^2 + x_2 - 3,\ f_2 = x_1 + 0.125x_2^2 - 1.5)$$

$(1,2)$ is a 3-fold solution. Form $P = (x_1 - 1, x_2 - 2)$, compute

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$$\dim(\mathbb{C}[\mathbf{x}]/(I,P^2)) = 2,$$
$$\dim(\mathbb{C}[\mathbf{x}]/(I,P^3)) = \dim(\mathbb{C}[\mathbf{x}]/(I,P^4)) = 3.$$

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So we have index $\rho = 3$, multiplicity $\mu = 3$.

Example 2 (continued)

The multiplication matrices (local ring) w.r.t. $\{x_1, x_2, 1\}$:

$$M_{x_1} = \begin{bmatrix} 0 & -1 & 3 \\ 6 & 3 & -10 \\ 1 & 0 & 0 \end{bmatrix}, \; M_{x_2} = \begin{bmatrix} 6 & 3 & -10 \\ -8 & 0 & 12 \\ 0 & 1 & 0 \end{bmatrix}.$$

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The primary component of I associated to $(1,2)$ is

$$(x_1^2 + x_2 - 3,\ x_2^2 + 8x_1 - 12,\ x_1x_2 - 6x_1 - 3x_2 + 10).$$

Differential Operators

- Let $D(\alpha) = D(\alpha_1, \ldots, \alpha_s) : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ denote the differential operator defined by:

$$D(\alpha_1, \ldots, \alpha_s) = \frac{1}{\alpha_1! \cdots \alpha_s!} \partial x_1^{\alpha_1} \cdots \partial x_s^{\alpha_s},$$

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- Let $\mathfrak{D} = \{D(\alpha) \,|\, |\alpha| \geq 0\}$, we define the space associated to I and $\hat{\mathbf{x}}$ as

$$\triangle_{\hat{\mathbf{x}}} := \{L \in Span_{\mathbb{C}}(\mathfrak{D}) \,|\, L(f)|_{\mathbf{x}=\hat{\mathbf{x}}} = 0, \ \forall f \in I\}.$$

Construct Differential Operators I

- Write Taylor expansion of $h \in \mathbb{C}[x]$ at $\hat{x}$:

$$T_{\rho-1}h(x_1,\ldots,x_s) = \sum_{\alpha\in\mathbb{N}^s, |\alpha|<\rho} c_\alpha (x_1-\hat{x}_1)^{\alpha_1}\cdots(x_s-\hat{x}_s)^{\alpha_s}.$$

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- Compute $\mathrm{NF}(h)$, and expand it at $\hat{\mathbf{x}}$

$$\mathrm{NF}(h(x)) = \sum_\beta d_\beta(\mathbf{x}-\hat{\mathbf{x}})^\beta,$$

and find scalars $a_{\alpha\beta} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\alpha\beta}c_\alpha$.

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  and find scalars $a_{\alpha\beta} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\alpha\beta} c_\alpha$.

- For each β such that $d_\beta \neq 0$, return

$$L_\beta = \sum_\alpha a_{\alpha\beta} \frac{1}{\alpha_1!\cdots\alpha_s!} \partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s} = \sum_\alpha a_{\alpha\beta} D(\alpha).$$

  $L = \{L_1,\ldots,L_\mu\}$ is a set of basis for $\triangle_{\hat{\mathbf{x}}}$.

  [Damiano, Sabadini, Struppa '07]

Example 2 (continued)

Write Taylor expansion at $(1,2)$ up to degree $\rho - 1 = 2$,

$$h(\mathbf{x}) = c_{0,0} + c_{1,0}(x_1 - 1) + c_{0,1}(x_2 - 2) + c_{2,0}(x_1 - 1)^2 \\ + c_{1,1}(x_1 - 1)(x_2 - 2) + c_{0,2}(x_2 - 2)^2.$$

Obtain the normal form of h by replacing x_1^2, x_1x_2, x_2^2 with

$$x_1^2 = -x_2 + 3, x_1x_2 = 6x_1 + 3x_2 - 10, x_2^2 = -8x_1 + 12.$$

The differential operators are:

$$\begin{cases} L_1 &= D(0,0), \\ L_2 &= D(0,1) - D(2,0) + 2D(1,1) - 4D(0,2), \\ L_3 &= D(1,0) - 2D(2,0) + 4D(1,1) - 8D(0,2). \end{cases}$$

Criterion of Involution of F_k

Let $\mathrm{T}_k(f_i) = \sum_{|\alpha|<k} f_{i,\alpha}(\mathbf{x}-\hat{\mathbf{x}})^{\alpha}$, and

$$F_k = \{\mathrm{T}_k(f_1),\ldots,\mathrm{T}_k(f_t),\ (x_1-\hat{x}_1)^{\alpha_1}\cdots(x_s-\hat{x}_s)^{\alpha_s},\ \sum\alpha_i = k\}.$$

Symbol matrices of F_k and prolongations are of full column rank. $M_k^{(j)}$ denotes coeff. matrices of the truncated prolonged system $\mathrm{T}_k(F^{(j)})$ with $\binom{k+s-1}{s}$ columns, $d_k^{(j)} = \dim \mathrm{Nullspace}(M_k^{(j)})$.

Theorem 6. *F_k is involutive at prolongation order m if and only if*

$$d_k^{(m)} = d_k^{(m+1)}$$

and $d_k = \dim(\mathbb{C}[\mathbf{x}]/(I,P^k)) = d_k^{(m)}$.

Compute Primary Component II

- Form the matrix $M_k^{(0)}$ by computing the truncated Taylor series expansions of $f_1, \ldots, f_t$ at $\hat{x}$ to order k. The prolonged matrix $M_k^{(j)}$ is computed by shifting $M_k^{(0)}$ accordingly.

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- Compute $d_k^{(j)} = \dim \text{Nullspace}(M_k^{(j)})$ for a given τ, until $d_k^{(m)} = d_k^{(m+1)} = d_k$.

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- If $d_k = d_{k-1}$, then set $\rho = k - 1$ and $\mu = d_\rho$.

Compute Primary Component II

- Form the matrix $M_k^{(0)}$ by computing the truncated Taylor series expansions of $f_1, \ldots, f_t$ at $\hat{x}$ to order k. The prolonged matrix $M_k^{(j)}$ is computed by shifting $M_k^{(0)}$ accordingly.

- Compute $d_k^{(j)} = \dim \text{Nullspace}(M_k^{(j)})$ for a given τ, until $d_k^{(m)} = d_k^{(m+1)} = d_k$.

- If $d_k = d_{k-1}$, then set $\rho = k - 1$ and $\mu = d_\rho$.

- Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ from the null vectors of $M_{\rho+1}^{(m)}$.

Example 3 [Leykin et al. 2006]

$$\{f_1 = x_1^3 + x_2^2 + x_3^2 - 1,\, f_2 = x_1^2 + x_2^3 + x_3^2 - 1,\, f_3 = x_1^2 + x_2^2 + x_3^3 - 1\}$$

has a 4-fold solution $\hat{\mathbf{x}} = (1,0,0)$. Transform it to the origin:

$$\begin{cases} g_1 &=& y_1^3 + 3y_1^2 + 3y_1 + y_2^2 + y_3^2, \\ g_2 &=& y_1^2 + 2y_1 + y_2^3 + y_3^2, \\ g_3 &=& y_1^2 + 2y_1 + y_2^2 + y_3^3. \end{cases}$$

has the 4-fold solution $\hat{\mathbf{y}} = (0,0,0)$. Let $I = (g_1, g_2, g_3)$, $P = (y_1, y_2, y_3)$.

$$[\mathrm{T}_3(g_1), \mathrm{T}_3(g_2), \mathrm{T}_3(g_3)]^T = M_3^{(0)} \cdot \left[y_1^2, \ldots, y_3, 1\right]^T,$$

$$M_3^{(0)} = \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$M_3^{(1)} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3 (continued)

- $d_3^{(0)} = 7$, $d_3^{(1)} = d_3^{(2)} = 4 \Longrightarrow d_3 = \dim(\mathbb{C}[\mathbf{y}]/(I, P^3)) = 4$.

- $d_4^{(0)} = 17$, $d_4^{(1)} = 8$, $d_4^{(2)} = d_4^{(3)} = 4$,
  $\Longrightarrow d_4 = \dim(\mathbb{C}[\mathbf{y}]/(I, P^4)) = 4$.

- $d_3 = d_4 = 4$, then index $\rho = 3$, multiplicity $\mu = 4$.

Construct Differential Operators II

Theorem 7. *Let* $Q = (I, P^{\rho})$ *be an isolated primary component of* I *at* $\hat{\mathbf{x}}$ *and* $\mu \geq 1$. *Suppose* $F_{\rho} = \mathrm{T}_{\rho}(F) \cup P^{\rho}$ *is involutive after* m *prolongations, the null space of the matrix* $M_{\rho}^{(m)}$ *is generated by* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\mu}$. *Then differential operators are:*

$$L_j = \mathbf{L} \cdot \mathbf{v}_j, \quad \text{for } 1 \leq j \leq \mu,$$

$$\mathbf{L} = [D(\rho - 1, 0, \ldots, 0), D(\rho - 2, 1, 0, \ldots, 0), \ldots, D(0, \ldots, 0)].$$

See also [Dayton and Zeng 2005].

Example 3 (continued)

Since $\rho = 3$, $\mu = 4$, and $d_3^{(0)} = 7$, $d_3^{(1)} = d_3^{(2)} = 4$, the null space of $M_3^{(1)}$ is:

$$N_3^{(1)} = [e_{10}, e_9, e_8, e_5],$$

Multiplying the diff. operators of order less than 3:

$$\{D(0,0,0),\ D(0,0,1),\ D(0,1,0),\ D(0,1,1)\}.$$

Example 1 (continued)

Since $\rho = 5$, $\mu = 11$, and

$$d_5^{(0)} = 23,\ d_5^{(1)} = d_5^{(2)} = 11,$$

Multiplying the diff. operators of order less than 5 w.r.t. to the null vectors of $M_5^{(1)}(35 \times 30)$, we get

$$D(0,0,0),\ D(1,0,0),\ D(0,0,1), D(0,1,0),$$
$$D(2,0,0),\ D(0,2,0), D(0,0,2),$$
$$D(0,0,3)+D(1,1,0),\ D(0,3,0)+D(1,0,1),\ D(3,0,0)+D(0,1,1),$$
$$D(0,0,4)+D(0,4,0)+D(4,0,0)+D(1,1,1)$$

Complexity for Computing Differential Operators

The complexity of our algorithm is:

$$O\left(t\binom{\rho+s-1}{s}^3\right).$$

The complexity of algorithm in [Mourrain MEGA'96] is:

$$O((s^2+t)\mu^3).$$

Notice $\mu \leq \binom{\rho+s-1}{s}$.

Approximate Singular Solution

- Suppose $\hat{\mathbf{x}}$ is an approximate singular solution of F:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\text{exact}} + \hat{\mathbf{x}}_{\text{error}}.$$

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- Construct multiplication matrices locally to refine the solution.

Refining Approximate Singular Solution(RASS)

- For approximate $\hat{\mathbf{x}}$ and tolerance τ, the prime ideal $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$, estimate μ and ρ.

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- Set $\hat{\mathbf{x}} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and run the first two steps for the refined solution and smaller τ.

- If $\hat{\mathbf{y}}$ converges to the origin, we get $\hat{\mathbf{x}}$ with high accuracy.

Example 3 (continued)

Given an approximate solution $\hat{\mathbf{x}} = (1.001, -0.002, -0.001\,i)$.

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Set $\tau = 10^{-2}$, we compute the singular solution of G:

$$\begin{aligned}\hat{\mathbf{y}} = (-0.0009994 - 7.5315 \times 10^{-10}\,i,\\ 0.002001 + 2.8002 \times 10^{-8}\,i,\\ -1.4949 \times 10^{-6} + 0.0010000\,i).\end{aligned}$$

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$$\begin{aligned}\hat{\mathbf{x}} = (&1 + 0.6\times 10^{-6} - 7.5315\times 10^{-10}\,i,\\ &0.1\times 10^{-5} + 2.8002\times 10^{-8}\,i,\\ &-1.4949\times 10^{-6}).\end{aligned}$$

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Apply twice for $\tau = 10^{-5}, 10^{-8}$ respectively, we get:

$$\begin{aligned}\hat{\mathbf{x}} = (1 + 7.0405 \times 10^{-18} - 7.8146 \times 10^{-19}\, i,\\ 1.0307 \times 10^{-16} - 1.9293 \times 10^{-17}\, i,\\ 1.5694 \times 10^{-16} + 7.9336 \times 10^{-17}\, i).\end{aligned}$$

Algorithm Performance

System	Zero	ρ	μ	RASS
cmbs1	$(0,0,0)$	5	11	$3 \to 11 \to 15$
cmbs2	$(0,0,0)$	4	8	$3 \to 13 \to 15$
mth191	$(0,1,0)$	3	4	$4 \to 9 \to 15$
LVZ	$(0,0,-1)$	7	18	$5 \to 10 \to 14$
KSS	$(1,1,1,1,1,1)$	5	16	$5 \to 11 \to 14$
Caprasse	$(2,-i\sqrt{3},2,i\sqrt{3})$	3	4	$4 \to 12 \to 15$
DZ1	$(0,0,0,0)$	11	131	$5 \to 14$
DZ2	$(0,0,-1)$	8	16	$4 \to 7 \to 14$
D2	$(0,0,0)$	5	5	$5 \to 10 \to 15$
Ojika1	$(1,2)$	3	3	$3 \to 6 \to 18$
Ojika2	$(0,1,0)$	2	2	$5 \to 10 \to 14$

Examples cited from `http://www.math.uic.edu/~jan/`,

[Dayton, Zeng '05, Dayton '07].

Thank you!

Grazie mille!