An introduction to Diagrammatic Specifications

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with *Christian Lair*

CIRM, 14 janvier 2004

This talk presents some
diagrammatic techniques
in computer science

In the field of categorical logic

with emphasis on sketches rather than categories
i.e., on specifications rather than complete theories

– I – Some basic examples

– II – Definitions and theorems

– III – An application to overloading

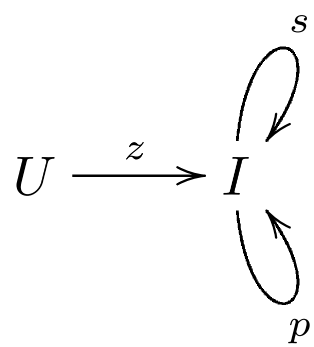
– I –

Some examples

In these examples, basically:

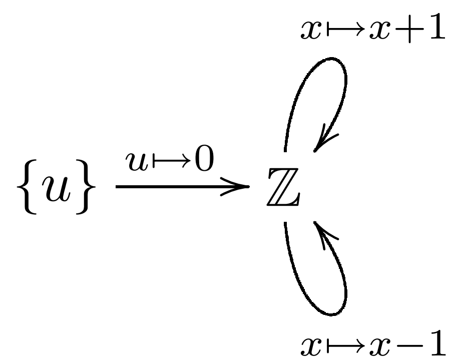
A specification S is
a directed graph, made of:

- points
  (vertices, sorts, types,...)
- arrows
  (edges, operations, functions,...)



A model M of S interprets:

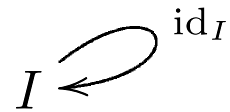
- points as sets
- arrows as maps



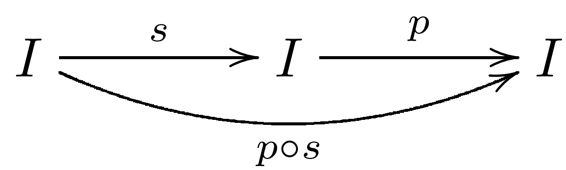
In these examples, actually:

A specification S is
a composition graph
i.e., a directed graph where:

- a point X
  can have an identity arrow
  $\mathrm{id}_X : X \to X$

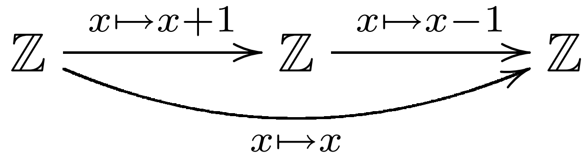


- a pair $(f : X \to Y, g : Y \to Z)$
  can have a composed arrow
  $g \circ f : X \to Z$



In a model M of S

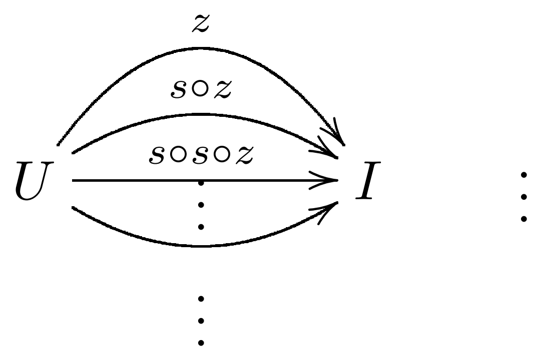
- an identity arrow becomes an identity map
- a composed arrow becomes a composed map



A specification S freely generates
a theory $F(S)$ with:

- all identity arrows
- all composed arrows
  (paths, terms,. . .)

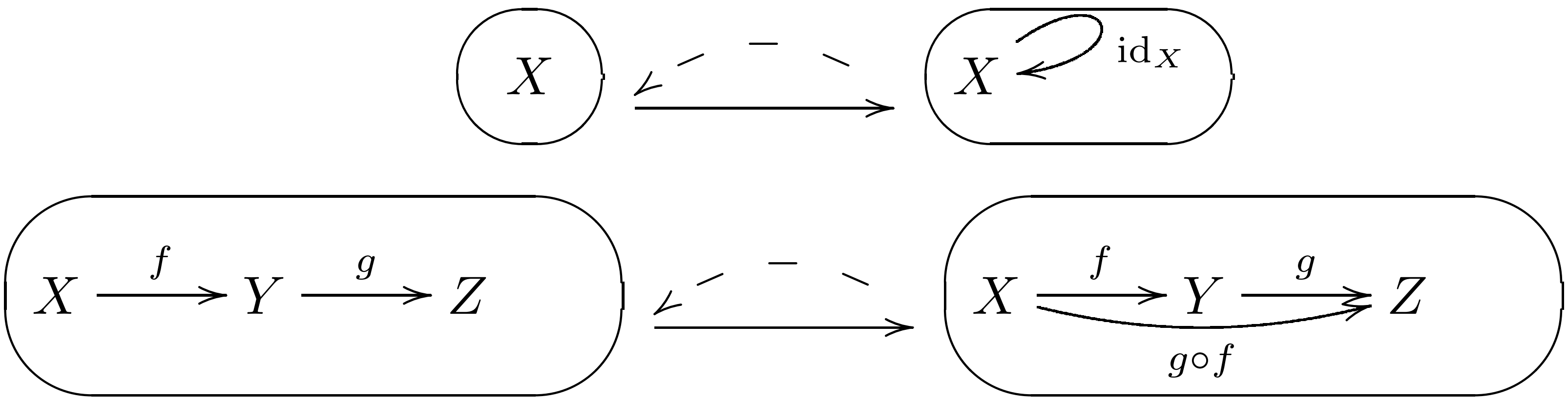
so that $F(S)$ is a category



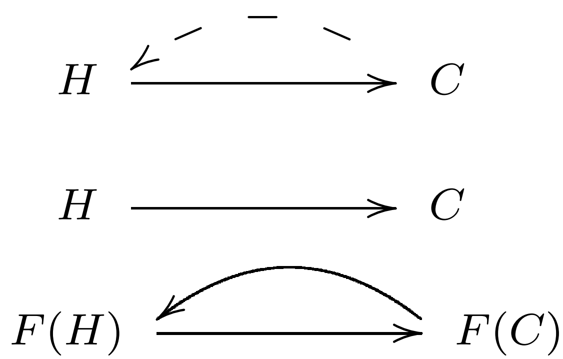
Fact (soundness)

$$\boxed{\mathrm{Mod}(S) = \mathrm{Mod}(F(S))}$$

The theory $F(S)$ is generated from the specification S by applying some deduction rules:

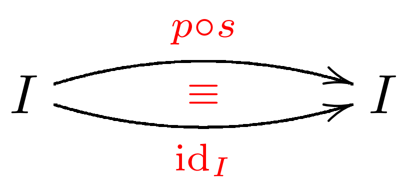


A deduction rule
is a morphism of specifications
which becomes
an isomorphism of theories



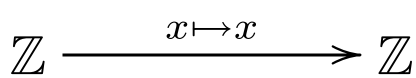
In addition

In a specification S
some pairs of parallel terms
can be called equations

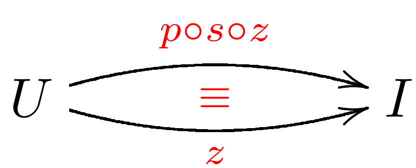


$p \circ s \equiv \mathrm{id}_I$

In a model M of S
equations become equalities



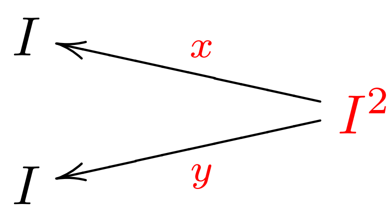
In the theory $F(S)$
equations generate congruences



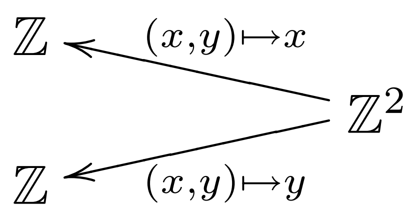
$p \circ s \circ z \equiv z$

Products

In a specification S
some projective cones
can be called potential products



In a model M of S
a potential product becomes
a cartesian product

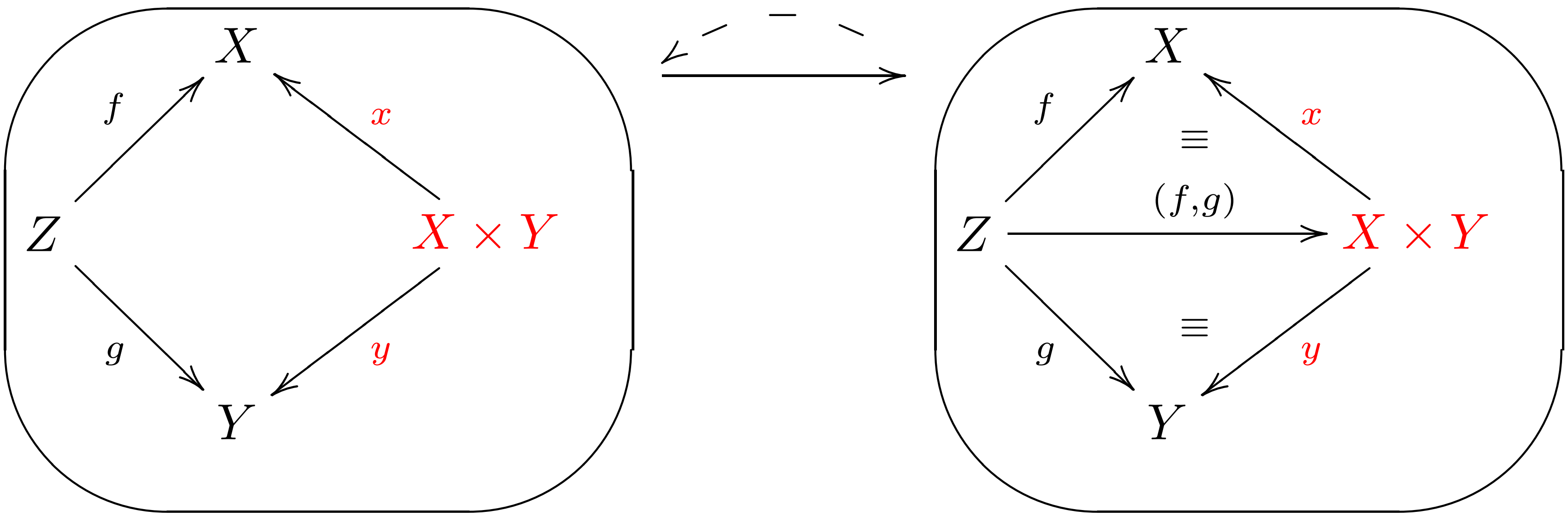


In the theory $F(S)$
a potential product becomes
an equiv-product

A potential product generates:

- new arrows (pairs)
- and new equations

according to the existence rule



and to the unicity rule

Constants

Besides binary products,
there can be n-ary products for any n, including $n = 0$.

In a specification S the vertex of
a nullary potential product $\qquad U$
is called a potential terminal point

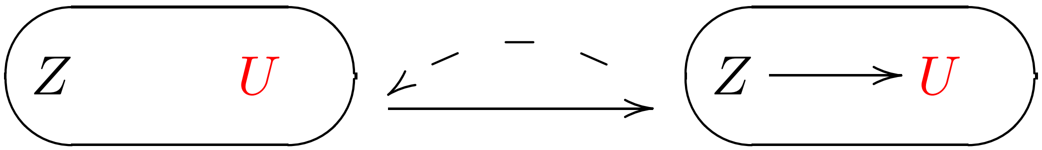
In a model M of S
it becomes a singleton $\qquad \{u\}$

Hence, a constant is an arrow from U $\qquad U \xrightarrow{z} I$

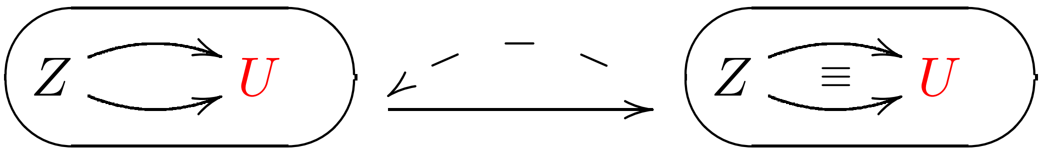
A potential terminal point generates:

- new arrows
- and new equations

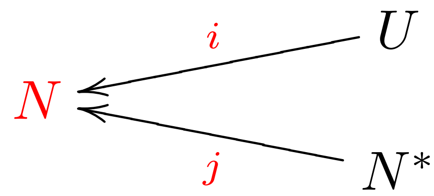
according to the existence rule



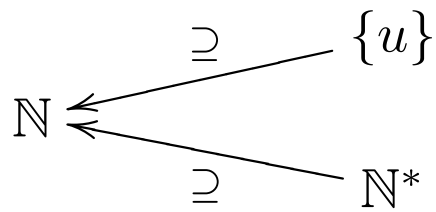
and to the unicity rule



Sums



In a specification S
some inductive cones
can be called potential sums



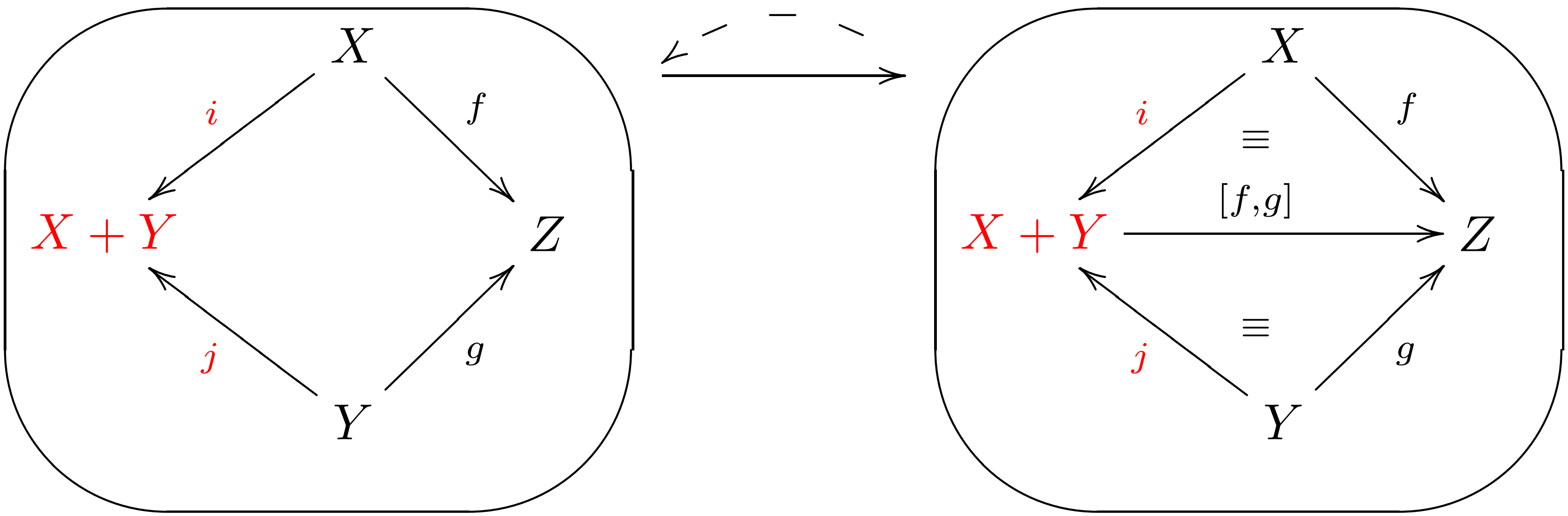
In a model M of S
a potential sum becomes
a disjoint union

In the theory $F(S)$
a potential sum becomes
an equiv-sum

A potential sum generates:

- new arrows (“if. . . then. . . else. . . ”)
- and new equations

according to the existence rule



and to the unicity rule

Initiality

In a specification S
some parts
can be called potentially initial

$$\begin{array}{c} U \\ z\big\downarrow \\ N \\ s\big\uparrow \\ N \end{array}$$

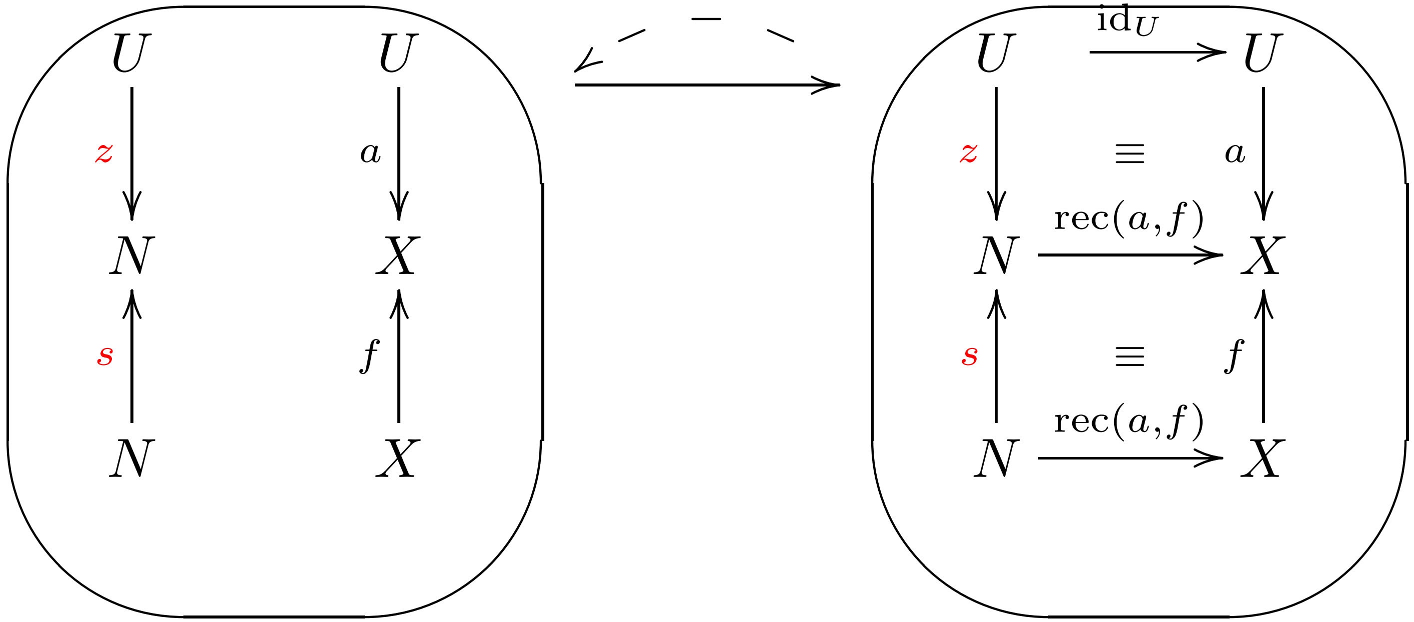
In a model M of S
potential initiality becomes
actual initiality

$$\begin{array}{c} \{u\} \\ u\mapsto 0\big\downarrow \\ \mathbb{N} \\ x\mapsto x+1\big\uparrow \\ \mathbb{N} \end{array}$$

A potential initiality generates:

- new arrows defined by induction
- and new equations proven by induction

according to the existence rule



and to the unicity rule

Terminality

In a specification S
some parts can be
called potentially terminal

$$\begin{array}{c} N \\ {\scriptstyle h}\big\uparrow \\ F \\ {\scriptstyle t}\big\downarrow \\ F \end{array}$$

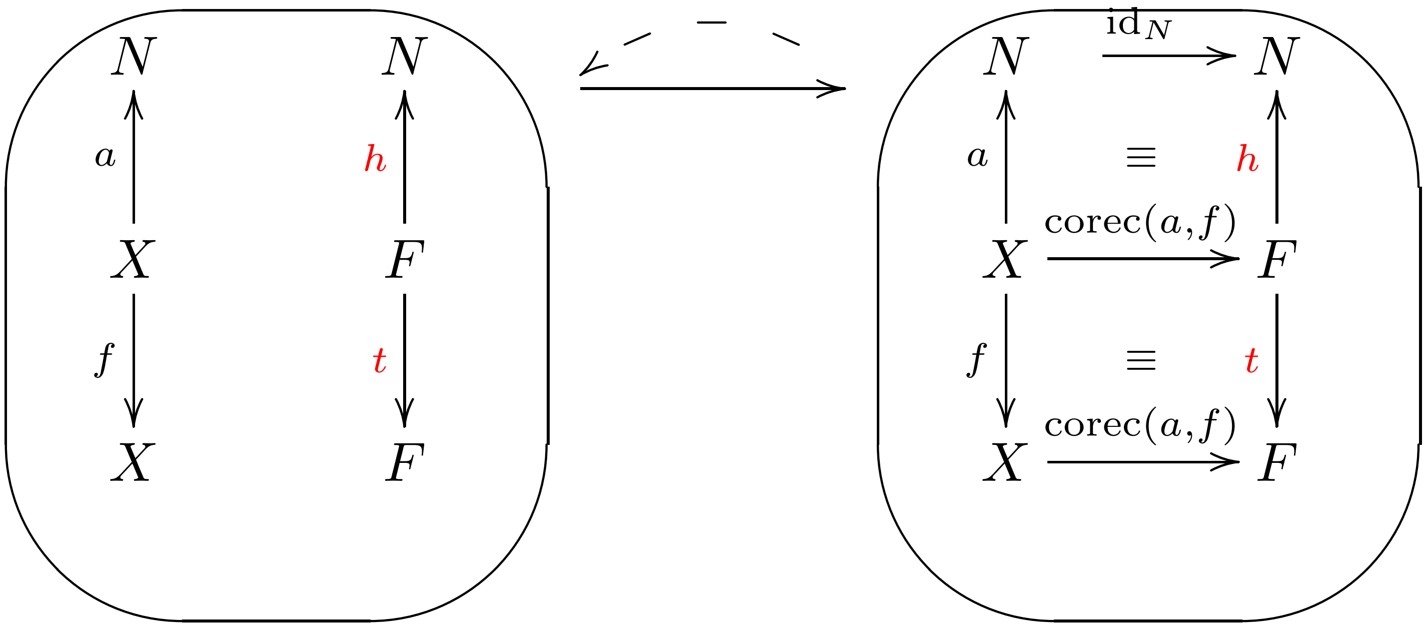
In a model M of S
potential terminality becomes
actual terminality

$$\begin{array}{r} \mathbb{N} \\ {\scriptstyle (x_0,x_1,\dots)\mapsto x_0}\big\uparrow \\ \mathbb{N}^\omega \\ {\scriptstyle (x_0,x_1,\dots)\mapsto(x_1,\dots)}\big\downarrow \\ \mathbb{N}^\omega \end{array}$$

A potential terminality generates:

- new arrows defined by coinduction
- and new equations proven by coinduction

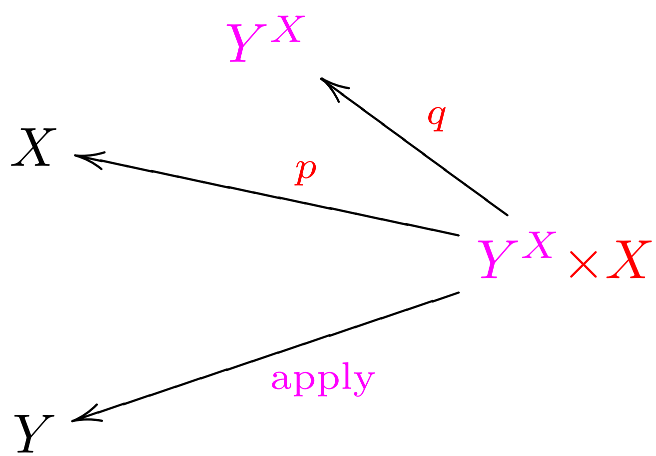
according to the existence rule



and to the unicity rule

Exponentials

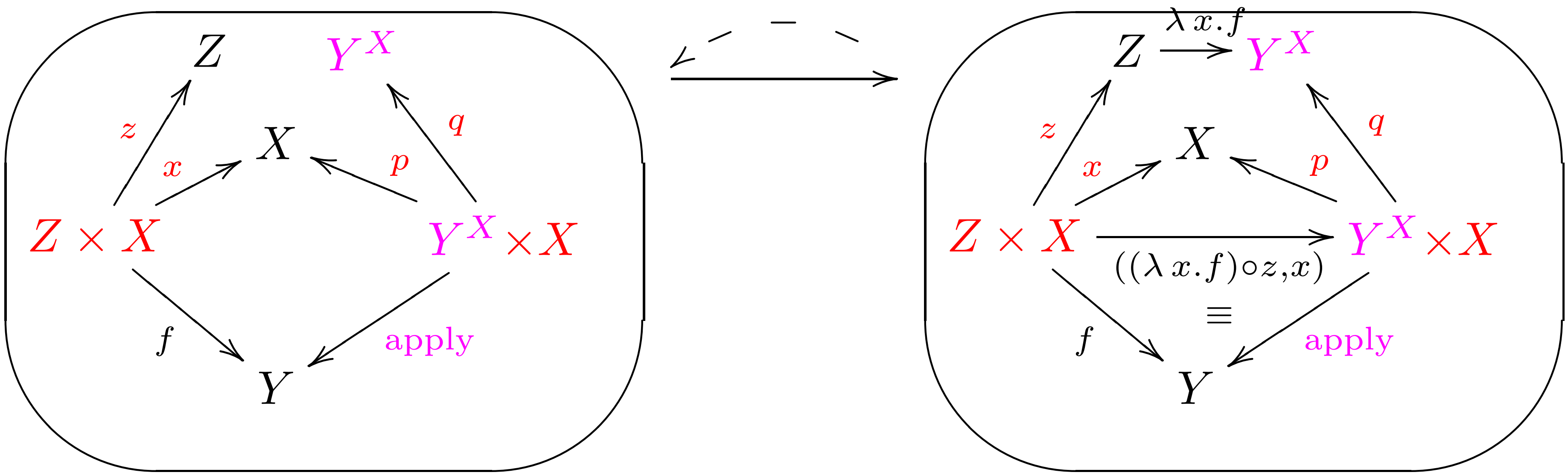
In a specification S
some parts can be called
potential exponentials



A potential exponential generates:

- new arrows (abstractions)
- and new equations (beta-equivalence)

according to the existence rule



and to the unicity rule

– II –

Definitions and theorems

Diagrammatic specifications generalize *Ehresmann*'s **sketches**:

1. a part of a specification S can be distinguished ("colored")
2. this results in constraints upon the models of S
3. and a related enrichment of the theory of S

For instance:

1. a projective cone of a specification S is colored
2. it must become a cartesian product in every model of S
3. and it gives rise to tuple of arrows in the theory of S

Definition

A **sketch** is a composition graph with

- potential limits (generalized products)
- potential colimits (generalized sums)

Theorem

$$\text{Sketches} \;\simeq\; \text{First-Order Logic}$$

Fact

Diagrammatic specifications generalize sketches

Definition

A **projective sketch** is a composition graph with

- potential limits (generalized products)

Theorem

$$\text{Projective Sketches} \;\simeq\; \text{Horn Clauses Logic}$$

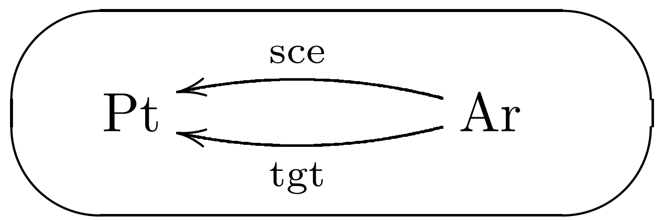
Fact

Diagrammatic specifications are defined from projective sketches

A basic example

The directed graphs are the models of the projective sketch

$\mathcal{E}_{\mathrm{Gr}} =$



All the specifications and theories in the previous examples
are directed graphs with additional features
i.e., they are models of projective sketches extending $\mathcal{E}_{\mathrm{Gr}}$

Propagators

Definition

A propagator is a morphism of projective sketches

$$P : \mathcal{E} \rightarrow \overline{\mathcal{E}}$$

For instance $P_{\text{Comp2Cat}} : \mathcal{E}_{\text{Comp}} \rightarrow \mathcal{E}_{\text{Cat}}$ where

- the models of $\mathcal{E}_{\text{Comp}}$ are the composition graphs
- the models of $\mathcal{E}_{\text{Cat}}$ are the categories
- P_{Comp2Cat} is the inclusion

"A propagator is some kind of "logical level" "

Basically

- **Meta-specification level**

A projective sketch : $\mathcal{E}_{\text{Comp}} = \quad \text{Pt} \underset{\text{tgt}}{\overset{\text{sce}}{\leftleftarrows}} \text{Ar} \ldots$

A model S of $\mathcal{E}$: $S = \quad \{U, N\} \underset{z \mapsto N, s \mapsto N}{\overset{z \mapsto U, s \mapsto N}{\leftleftarrows}} \{z, s\} \ldots$

- **Specification level**

S is also a specification : $S = \quad U \xrightarrow{z} N \circlearrowleft^{s}$

A model M of S : $M = \quad \{u\} \xrightarrow{u \mapsto 0} \mathbb{N} \circlearrowleft^{x \mapsto x+1}$

The definitions

With respect to a propagator

$$\boxed{P : \mathcal{E} \to \overline{\mathcal{E}}}$$

- A P-specification S is a model of $\mathcal{E}$

$$\boxed{\mathrm{Spec}(P) = \mathrm{Mod}(\mathcal{E})}$$

- A P-domain Δ is a model of $\overline{\mathcal{E}}$

$$\boxed{\mathrm{Dom}(P) = \mathrm{Mod}(\overline{\mathcal{E}})}$$

A P-domain Δ has an underlying P-specification $G(\Delta)$

- A P-model of S with values in Δ is a morphism

$$\boxed{M : S \to G(\Delta)}$$

For instance, with respect to the propagator

$$P_{\text{Comp2Cat}} : \mathcal{E}_{\text{Comp}} \to \mathcal{E}_{\text{Cat}}$$

- A P-specification S is a composition graph
- A P-domain Δ is a category

  A category Δ has an underlying composition graph $G(\Delta)$
- A P-model of S with values in Δ is a functor $M : S \to G(\Delta)$

When $\quad S = U \xrightarrow{z} N \circlearrowleft^{s} \quad$ and $\quad \Delta = \text{Set}$

one P-model M of S with values in Δ is such that:

$$M(S) = \{u\} \xrightarrow{u \mapsto 0} \mathbb{N} \circlearrowleft^{x \mapsto x+1}$$

Adjunction

Theorem (*Ehresmann*)
The omitting functor $G : \mathrm{Dom}(P) \to \mathrm{Spec}(P)$
has a left-adjoint $F : \mathrm{Spec}(P) \to \mathrm{Dom}(P)$

$$\mathrm{Spec}(P) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Dom}(P)$$

$F(S)$ is freely generated by S, i.e., $F(S)$ is the theory of S

Theorem (adjunction)
A P-model of S with values in Δ is a morphism

$$\boxed{M : S \to G(\Delta)} \text{ or, equivalently, } \boxed{M : F(S) \to \Delta}$$

A note on dynamic evaluation

This result is **false** when $P : \mathcal{E} \to \overline{\mathcal{E}}$
for **non-projective** sketches $\mathcal{E}$ and $\overline{\mathcal{E}}$

However, several theorems by *Guitart and Lair* generalize it

When $\mathcal{E}$ and $\overline{\mathcal{E}}$ have sums (but no general colimits),
there is a discrete family of P-domains
"instead of" just one P-domain $F(S)$

For instance, there is one initial ring (with unit) $\mathbb{Z}$,
but several locally initial fields : the prime fields $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$,..., $\mathbb{Q}$.

Decomposition of a propagator

The propagator

$$P_0 = P_{\mathrm{Gr2Cat}} : \mathcal{E}_{\mathrm{Gr}} \to \mathcal{E}_{\mathrm{Cat}}$$

is such that

$$G_0(F_0(G)) \neq G \quad \text{and} \quad F_0(G_0(C)) \neq C$$

Fact

The propagator P_0 can be decomposed as $P_0 = P_2 \circ P_1$

$$\begin{array}{ccccc} \mathcal{E}_{\mathrm{Gr}} & & \xrightarrow{\quad P_0 \quad} & & \mathcal{E}_{\mathrm{Cat}} \\ & & = & & \\ & \searrow_{P_1} & \mathcal{E}_{\mathrm{Comp}} & \nearrow_{P_2} & \end{array}$$

where

$$G_1(F_1(G)) \cong G \quad \text{and} \quad F_2(G_2(C)) \cong C$$

Decomposition of all propagators

Theorem *(Duval-Lair)*
Every propagator $P : \mathcal{E} \to \overline{\mathcal{E}}$ can be decomposed as $P = P_2 \circ P_1$

$$\begin{array}{ccccc} \mathcal{E} & & \xrightarrow{\quad P \quad} & & \overline{\mathcal{E}} \\ & \searrow^{P_1} & = & \nearrow^{P_2} & \\ & & \mathcal{E}' & & \end{array}$$

where

F_1 "trivial" and $G_1 \circ F_1 \cong \mathrm{id}_{\mathcal{E}}$ and $F_2 \circ G_2 \cong \mathrm{id}_{\overline{\mathcal{E}}}$

Theorem (*Hébert-Adámek-Rosický*)
The propagator $P_2 : \mathcal{E}' \to \overline{\mathcal{E}}$ is (essentially) made of
the inversion of some arrows

Deduction rules

Definition

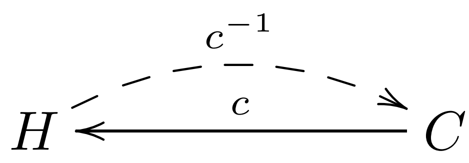
A fractioning propagator $P : \mathcal{E} \to \overline{\mathcal{E}}$ is such that

$$F \circ G \cong \mathrm{id}_{\overline{\mathcal{E}}}$$

i.e., P is made of the inversion of some arrows
(for instance $P_{\text{Comp2Cat}} : \mathcal{E}_{\text{Comp}} \to \mathcal{E}_{\text{Cat}}$)

Illustration

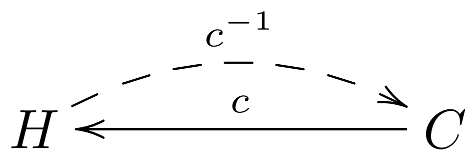
A fractioning propagator P is illustrated by a copy of $\mathcal{E}$ together with a dashed arrow for each inverse added in $\overline{\mathcal{E}}$:



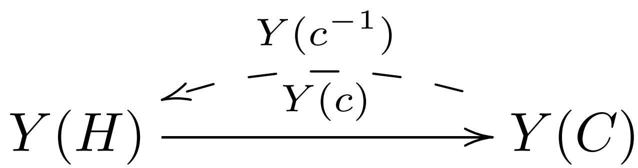
Then H =hypotheses and C =conclusion and c^{-1} = the rule $\frac{H}{C}$

Deduction rules : the Yoneda functor

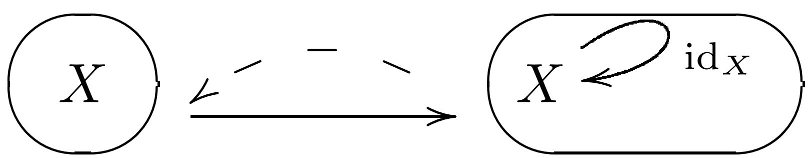
The Yoneda contravariant functor for projective sketches (*Lair*) maps



to P-specifications



For instance



– III –

An application to overloading

with *Hélène Kirchner* and *Christian Lair*

Now, for simplicity, a specification is just
a composition graph (with equations)

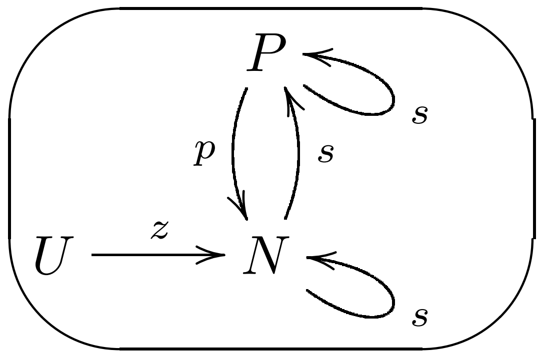
Overloading occurs in a specification
when several arrows share the same name

In order to make a clear distinction between an arrow and its name,
the names are considered as arrows in another specification, and
the fact of naming the arrows as a morphism of specifications.

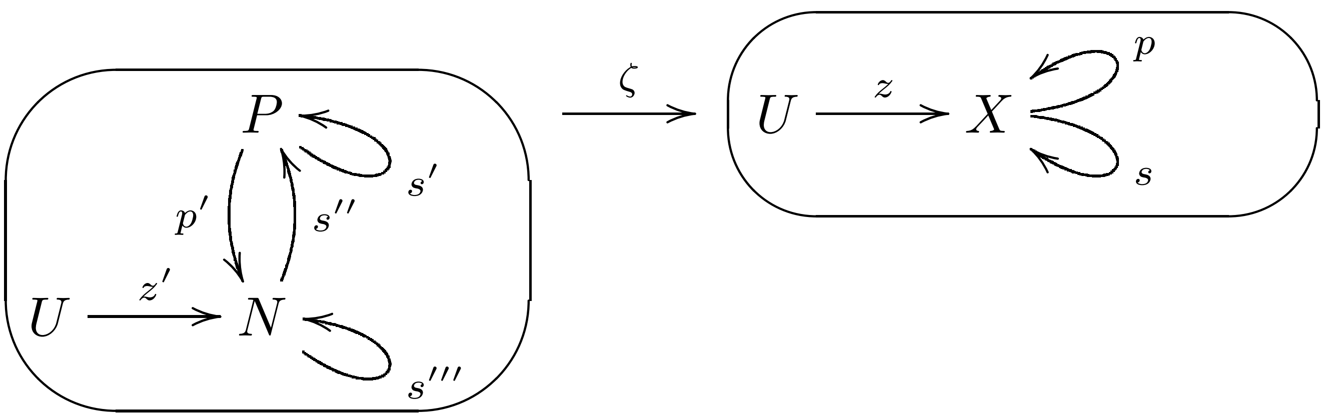
Definition

An overloaded specification is a morphim of specifications

For instance, the overloaded specification

$Z =$ 

is considered as a morphism $\zeta : T \to S$

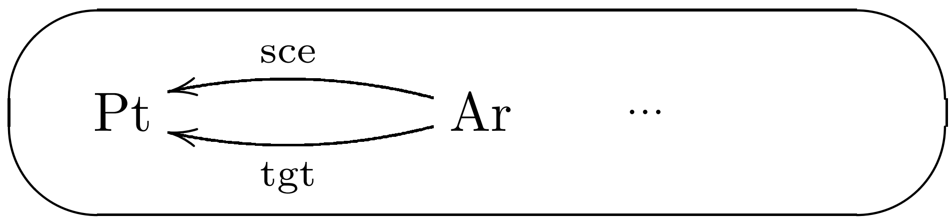


Static type-checking is done in S : $p \circ z$ is accepted

Dynamic type-checking is done in T : $p \circ z$ is rejected

A (not-overloaded) specification is a model of

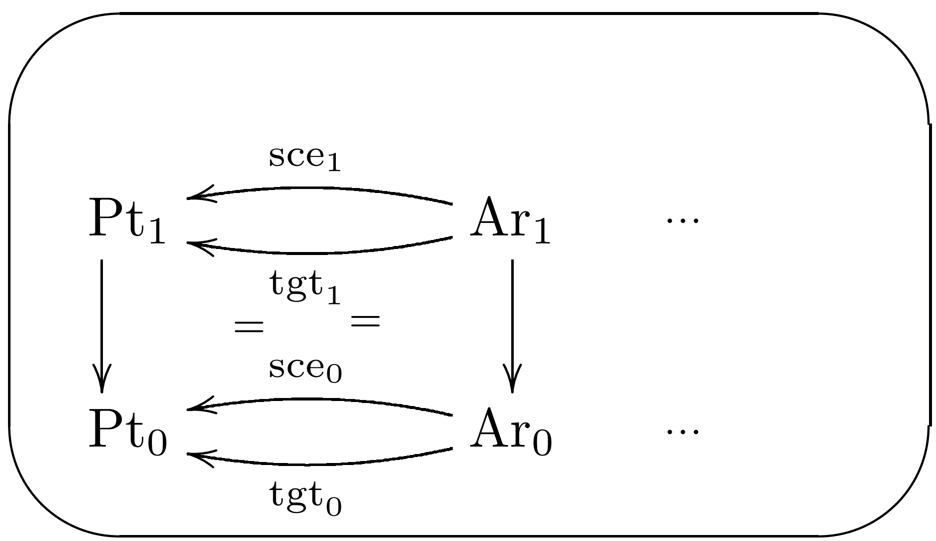
$\mathcal{E}_{\mathrm{Comp}} =$



Proposition

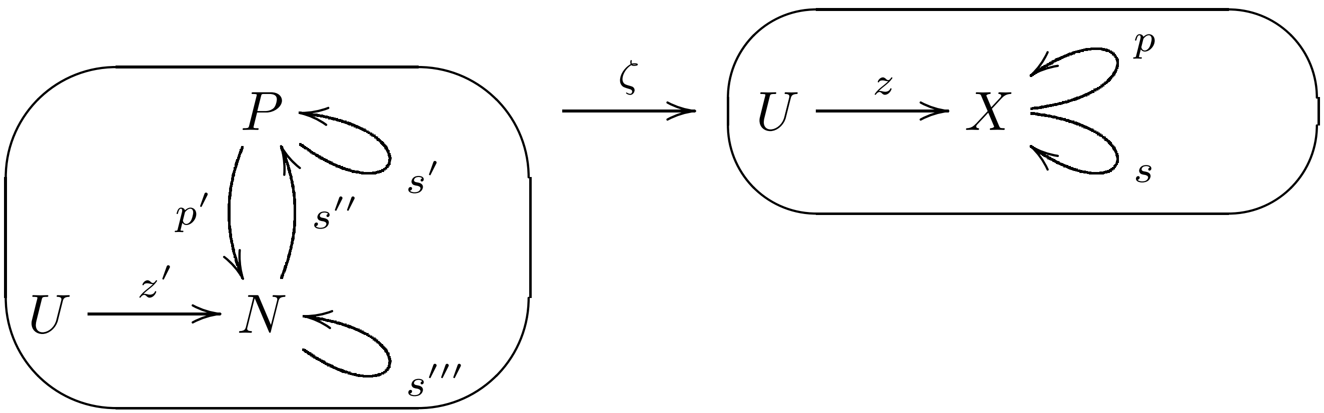
An overloaded specification is a model of

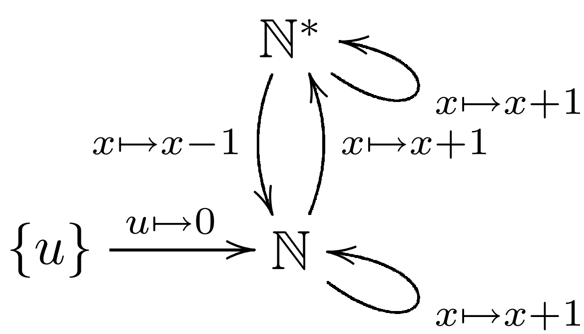
$\mathcal{E} =$



What are the models of an overloaded specification $\zeta : T \to S$?

Naive view : $\boxed{\mathrm{Mod}(\zeta) = \mathrm{Mod}(T)?}$





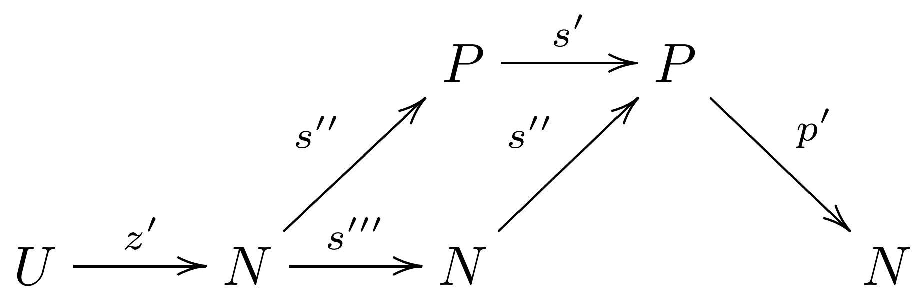
One semantics of overloading

Identification rule :

two arrows with the same name, source and target,

must have the same interpretation

For instance $p \circ s \circ s \circ z : U \rightarrow N$



What are the models of an overloaded specification $\zeta : T \to S$?

$\zeta : T \to S$ generates $F(\zeta) : T' \to F(S)$
where $T' \neq F(T)$ because of the identification rule

Definition

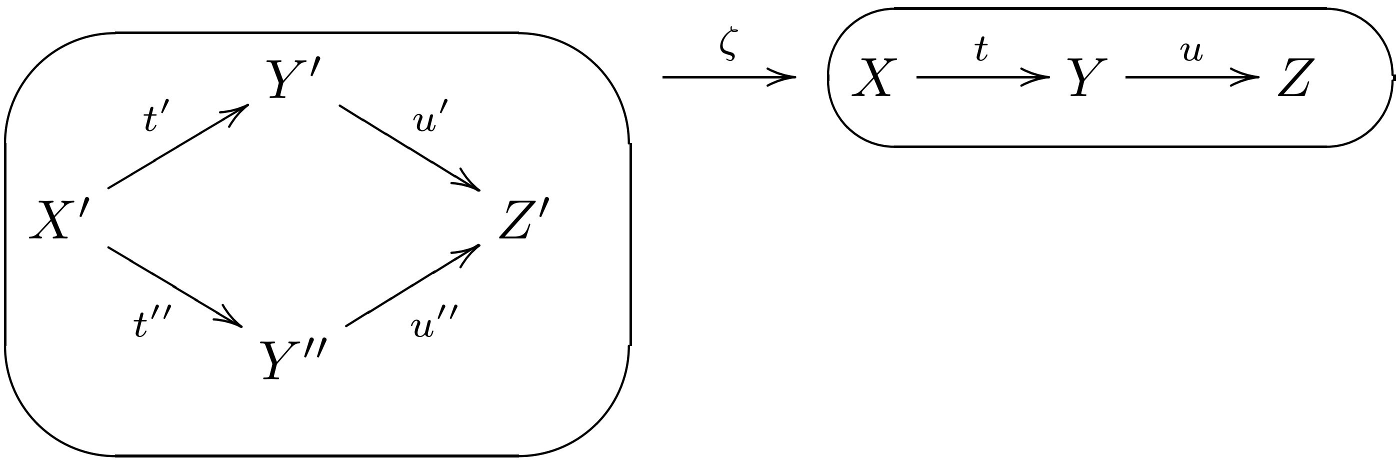
$$\boxed{\mathrm{Mod}(\zeta) = \mathrm{Mod}(T')}$$

Proposition ("non-soundness")

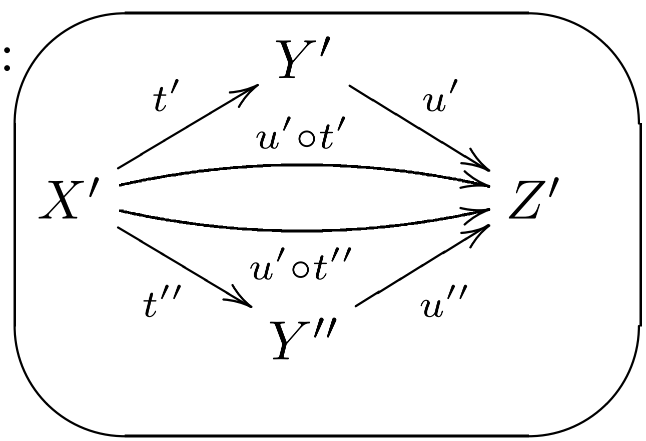
$$\boxed{\mathrm{Mod}(T) \neq \mathrm{Mod}(\zeta)}$$

The semantics of T is an approximation of the semantics of ζ

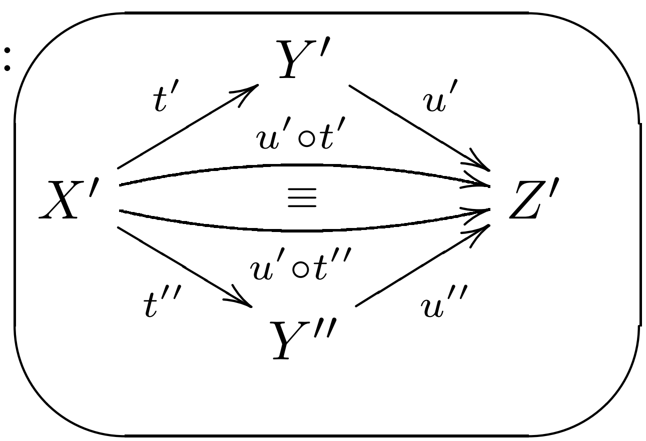
The diamond example



in $F(T)$:



in T':



Conclusion

Diagrammatic specifications are both
simple and **powerful**

They are not restricted to graph-based specifications
(cf. applications to overloading
with *H. Kirchner and C. Lair*)

Specifications with respect to several propagators
can easily be mixed together
(cf. applications to zooming
with *C. Lair, C. Oriat and J.-C. Reynaud*)