Induction over real numbers

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Join work with Gilles Dowek

Real Analysis

An elementary lemma :

P closed subset of $\mathbb{R}$

$P(c)$ *Initialization*

$\forall x \in \mathbb{R}, P(x) \Rightarrow [\ \exists\, \varepsilon > 0\ P([x, x+\varepsilon[)\ ]$ *Heredity*

$\forall x \in [c, +\infty[\ P(x)$ $\hookrightarrow$ *Universality*

Principle of «closed» induction

A useful tool

Differential equations (kinematics, oscillators...):

- Existence of a unique maximal solution in Cauchy-Lipschitz theorem
- Qualitative study of the solutions of a differential equation
- . . .

A «dual» property

A open set of $[0,1]$

$\forall x \in [0,1], \ [\forall y < x, y \in A] \Rightarrow x \in A$ *Heredity*

$$\frac{\forall x \in [0,1], \ [\forall y < x, y \in A] \Rightarrow x \in A}{\forall x \in [0,1], x \in A}$$

Principle of open induction

Links with completeness?

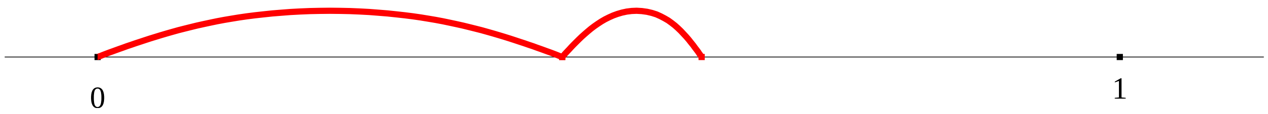
Closed Induction $\Rightarrow$ Upper Bound Property

Closed Induction $\Leftrightarrow$ Open Induction

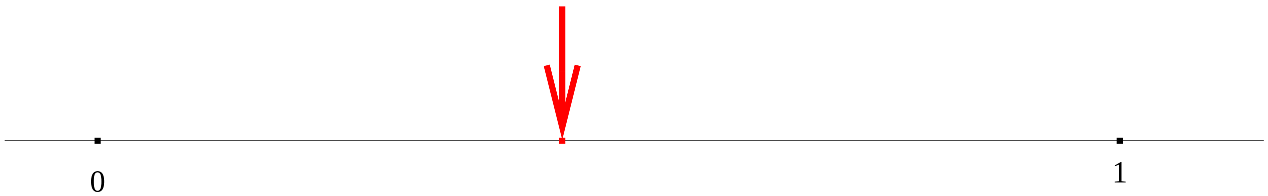
Computational content of these principles ?
Focus on the open induction principle

Achille's run

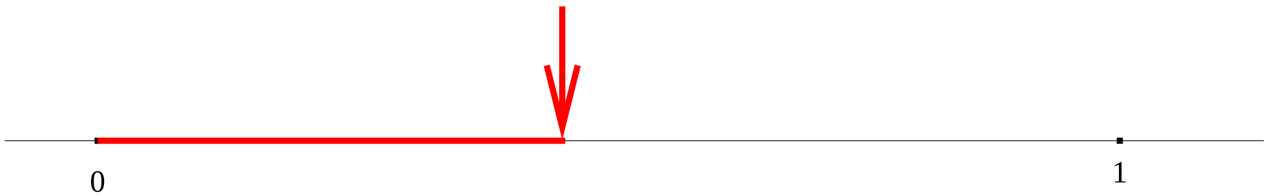
Achille's run



Achille's run



Achille's run



Achille's run

Intuitionism

Th. Coquand's proof	*W. Veldman's proof*
Working with $\{0, 1\}^\omega$	Working with constructive reals
Open sets of $\{0, 1\}^\omega$	Enumerative open sets
Bar induction	Almost fan theorem

No intuitionistic proof without an extra axiom?

Two kind of difficulties

- To choose the extra axiom:
  - Intuitionistic axiom(!)
  - Goal = realization of the OI principle

Bar Induction

- To choose the open sets :
  - Non equivalence in intuitionistic logic
  - Definition fitted to the proof
  - Definition fitted to realization

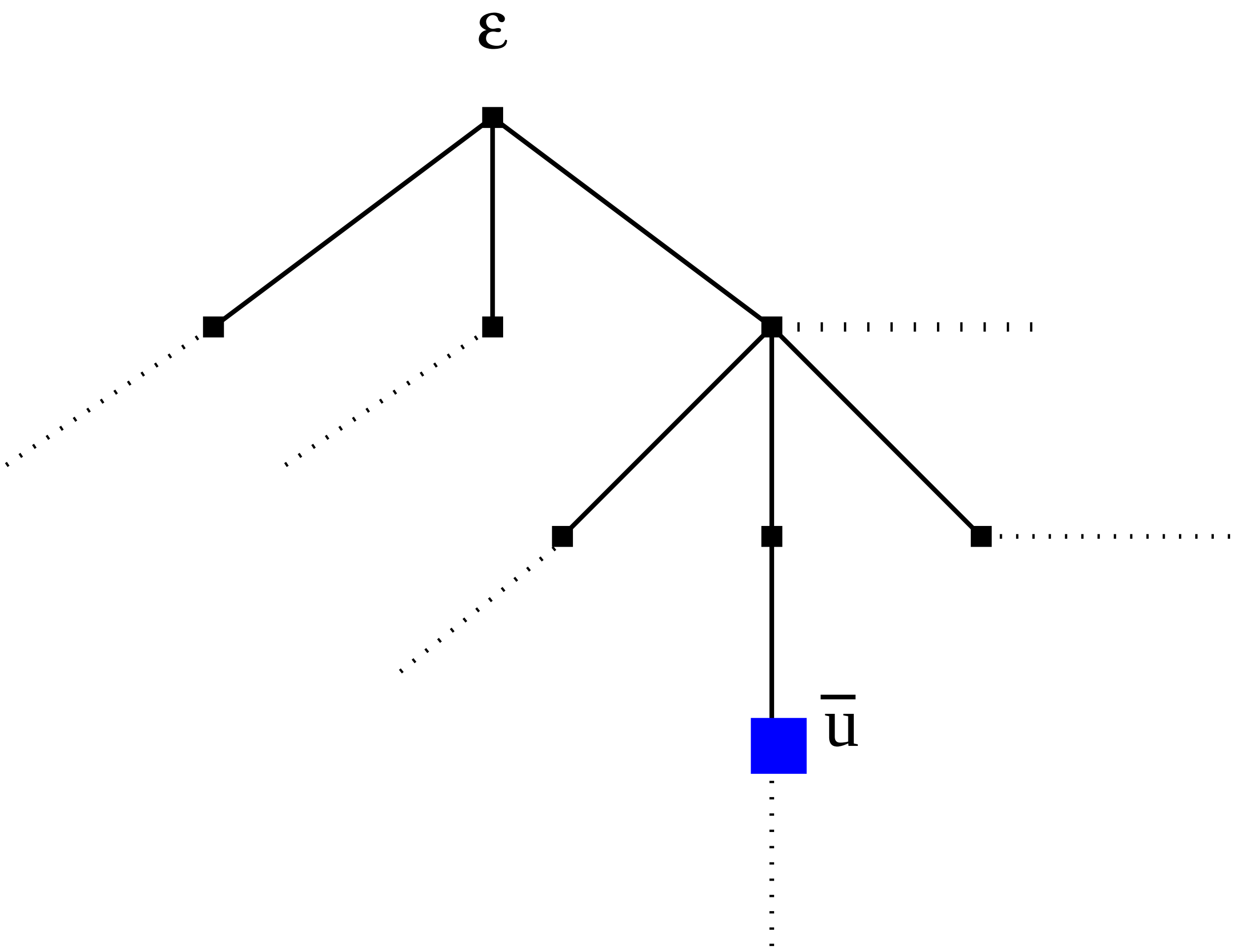
Enumerative open sets

Bars

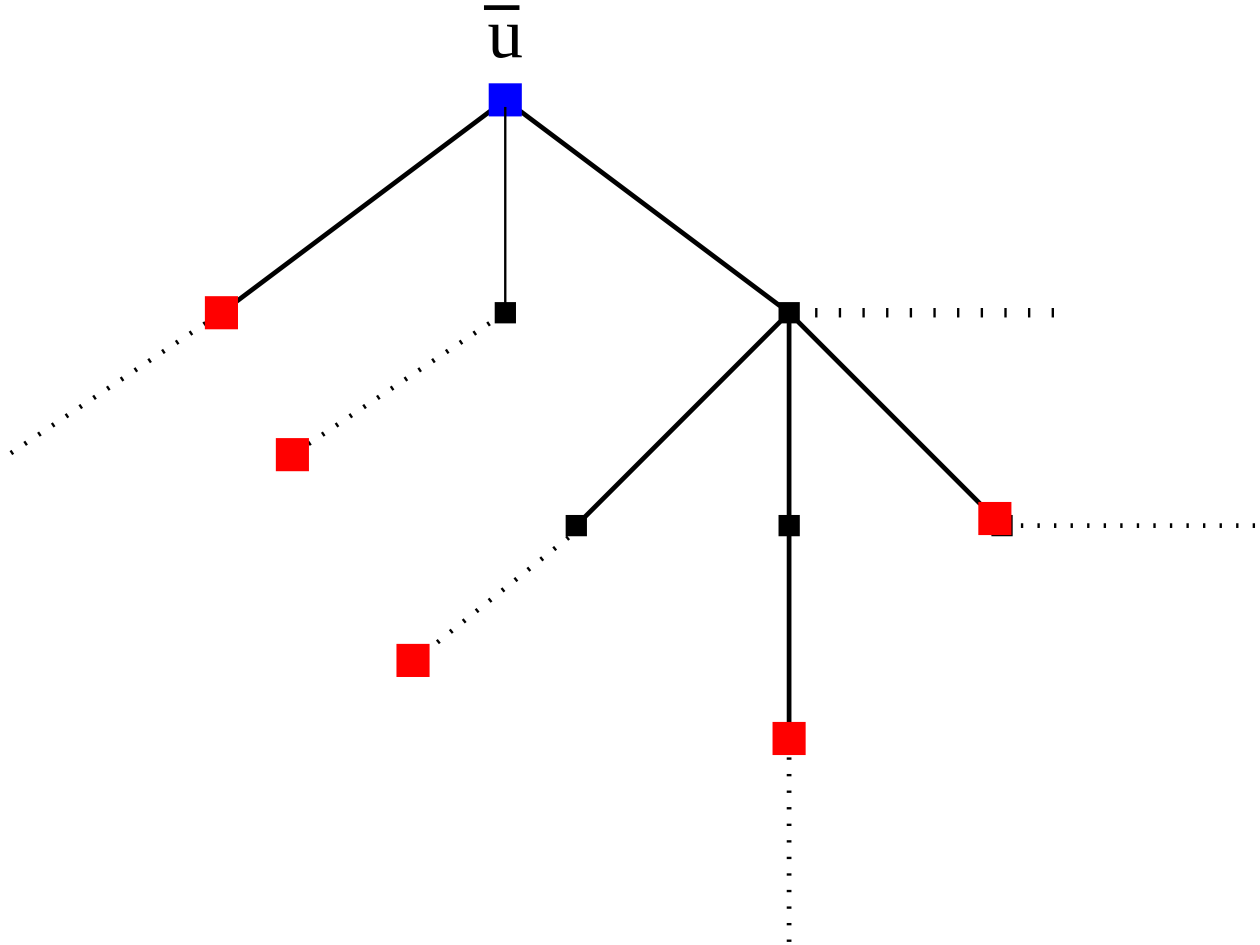
- Tree structure with countable branching : $\mathbb{N}^*$
- $\overline{u} = u_1 \ldots u_l \in \mathbb{N}^*$
- X predicate over $\mathbb{N}^*$
- $X|\overline{u}$:

$$\forall\ \overline{u}n_1 \ldots n_k \cdots \in \mathbb{N}^\omega\ \exists k_0 \text{ st. } X(\overline{u}n_1 \ldots n_{k_0})$$

Picture of a bar



Picture of a bar

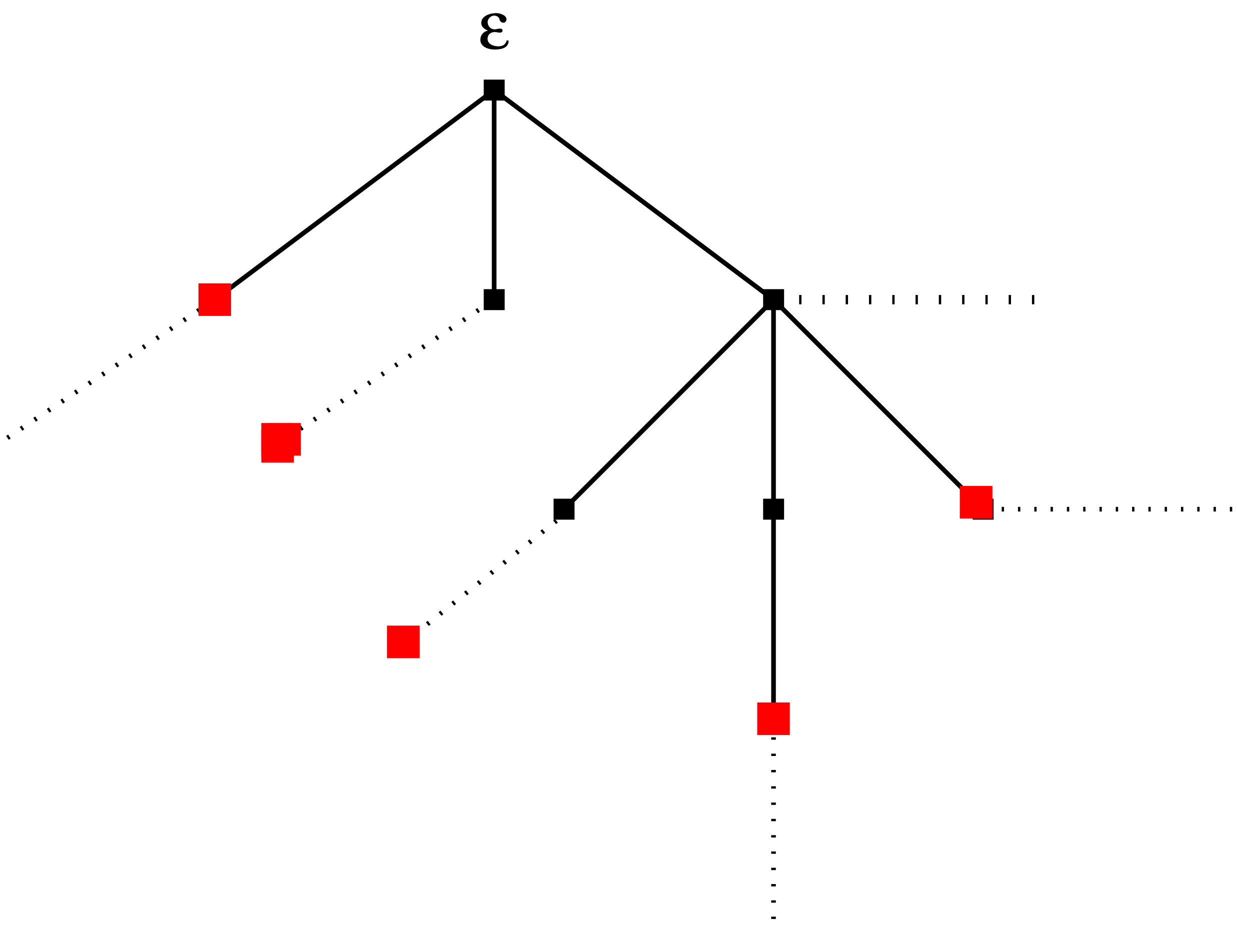


Bar induction axiom

X and Y predicates over $\mathbb{N}^*$

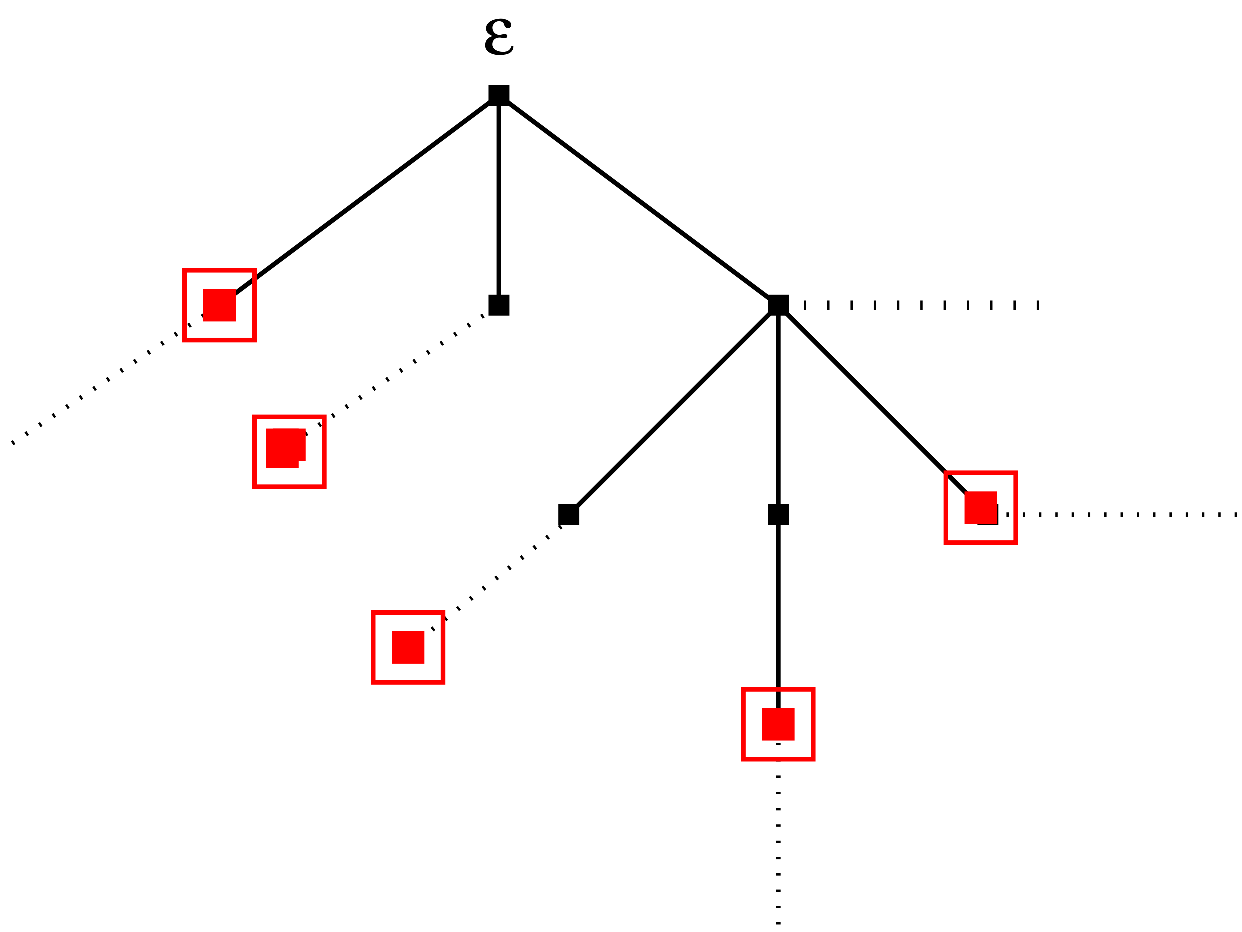
$$\frac{\begin{array}{lr} \forall \overline{u} \in \mathbb{N}^* \; [X(\overline{u}) \Rightarrow Y(\overline{u})] & X \text{ included in } Y \\ [\forall \overline{u} \in \mathbb{N}^* \; \forall a \in \mathbb{N} \; [X(\overline{u}) \Rightarrow X(\overline{u} \bullet a)] & X \text{ is monotonous }] \\ \forall \overline{u} \in \mathbb{N}^* \; [\forall a \in \mathbb{N} \; Y(\overline{u} \bullet a)] \Rightarrow Y(\overline{u}) & Y \text{ is hereditary} \\ X | \overline{u} & X \text{ bars } \overline{u} \end{array}}{Y(\overline{u})}$$

Bar induction



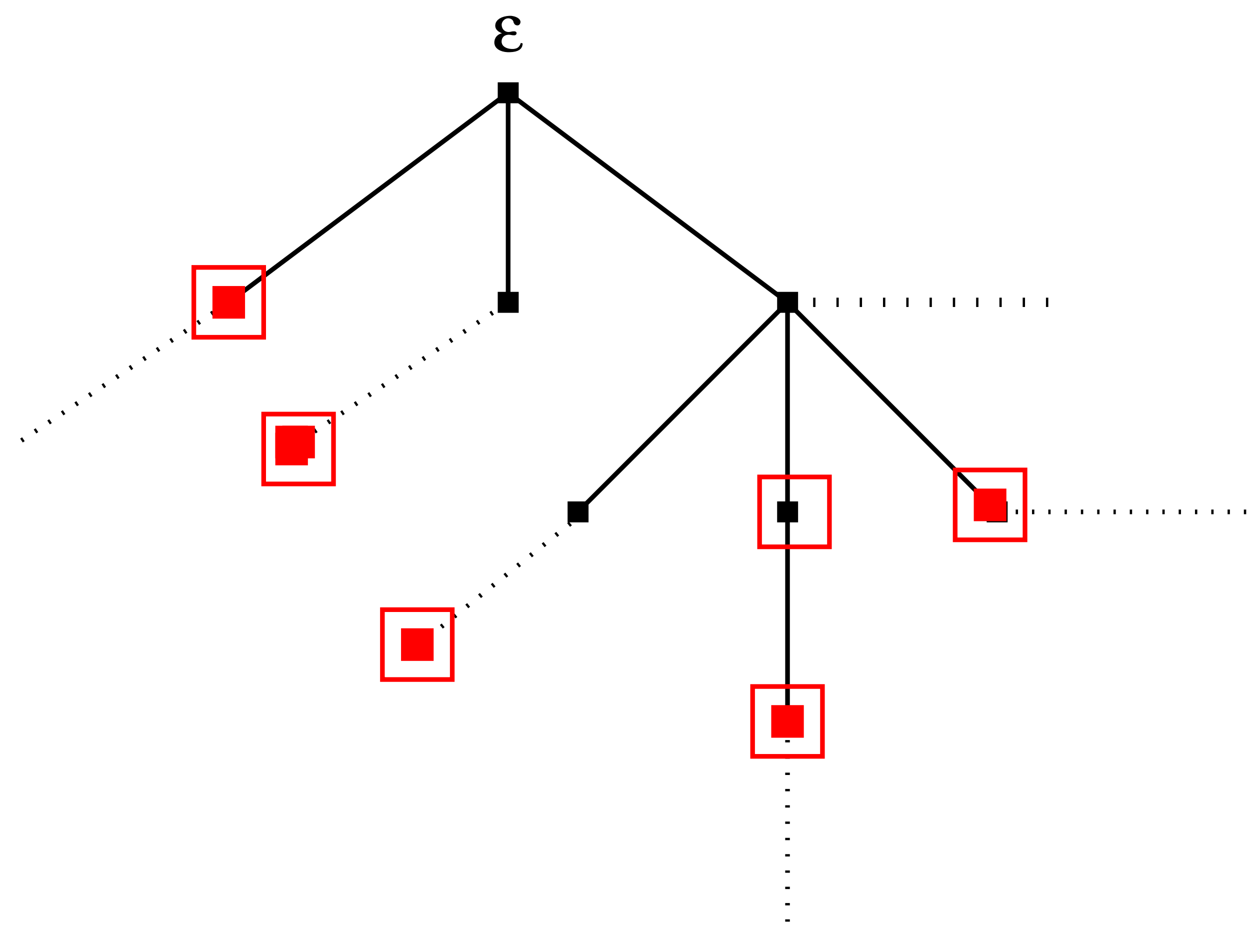
: nodes belonging to the bar X

Bar induction



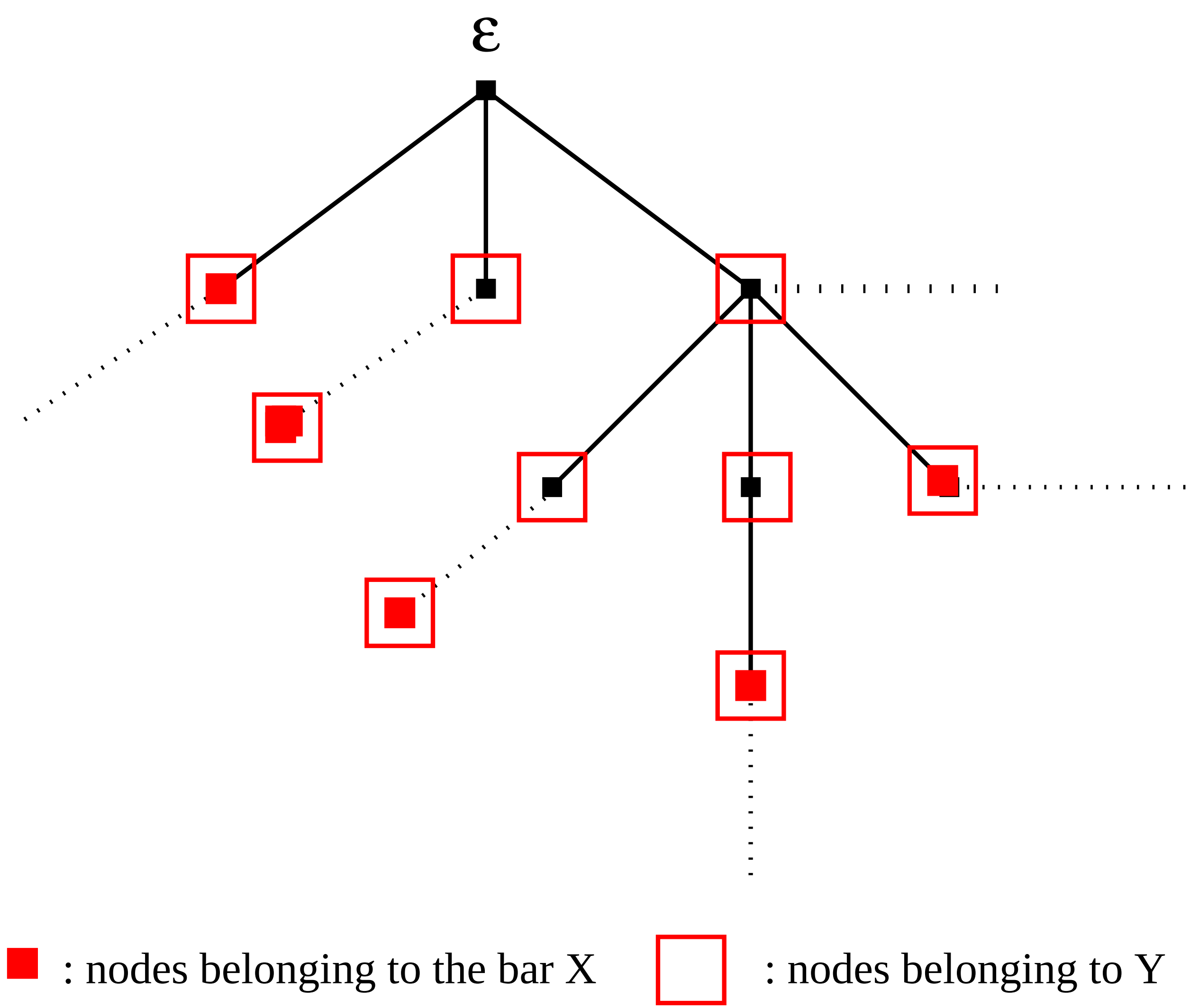
: nodes belonging to the bar X : nodes belonging to Y

Bar induction

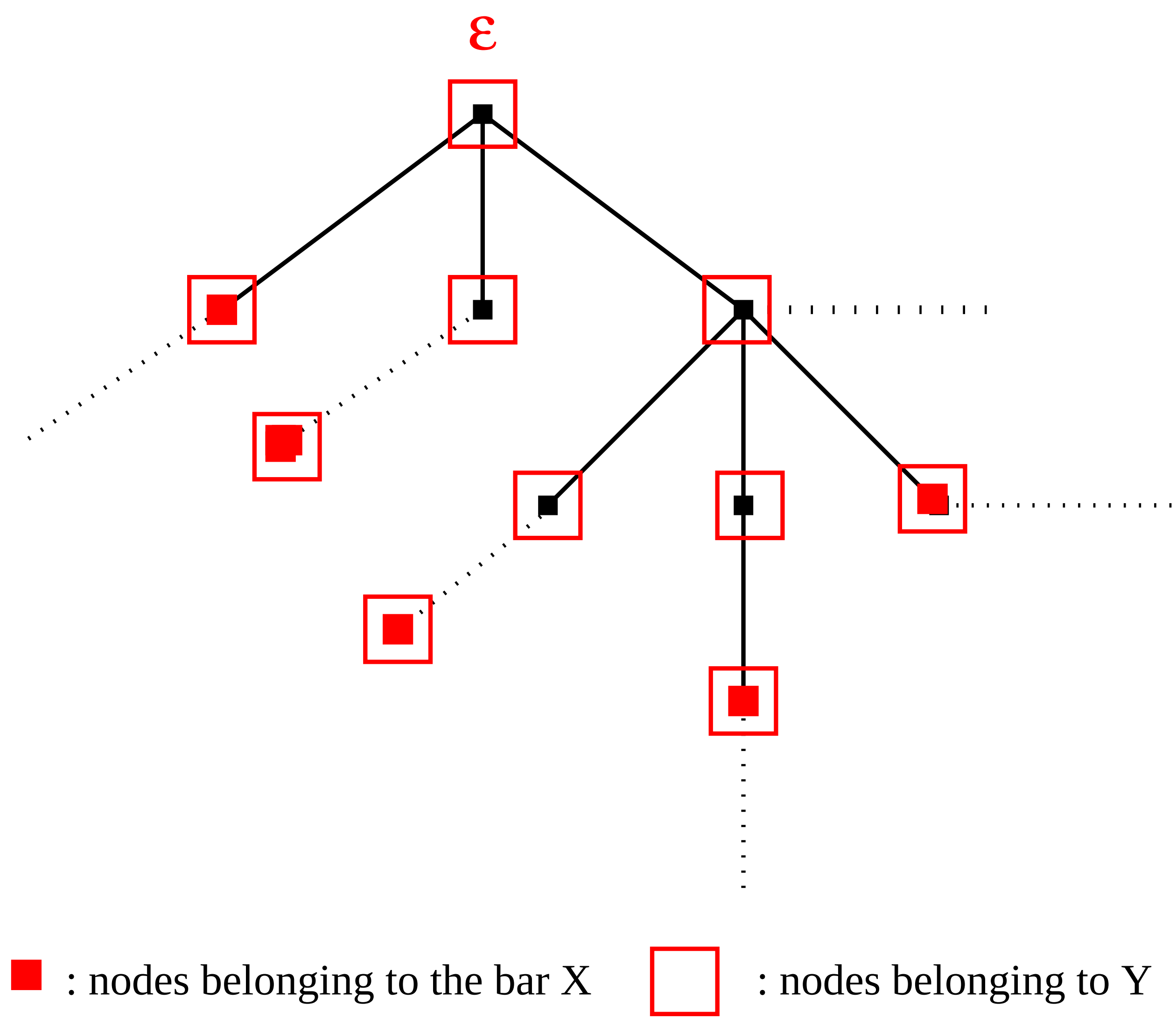


: nodes belonging to the bar X   : nodes belonging to Y

Bar induction



Bar induction



Constructive open sets

Three ways of considering real open sets :

- Inverse image by the canonical surjection $\{0,1\}^\omega \to \mathbb{R}$
- Neighbourhood : $\forall\, x \in A$, provide ε so that $]x-\varepsilon, x+\varepsilon[\subseteq A$
- Give an enumeration of open intervals :
$$A = \bigcup_{i\in\mathbb{N}}]\alpha_i, \beta_i[, \quad \forall i \in \mathbb{N}, \alpha_i, \beta_i \in \mathbb{Q}$$

«Enumerative» definition

Complete statement of the OI

- A is an «enumerative» open set of $[0,1]$:

$$g : i \in \mathbb{N} \mapsto]\alpha_i, \beta_i[$$

- A is «inductive»:

$$\forall x \in [0,1],\ [\forall y < x, y \in A] \Rightarrow x \in A \quad (*)$$

Then $A = [0,1]$

Tree encoding

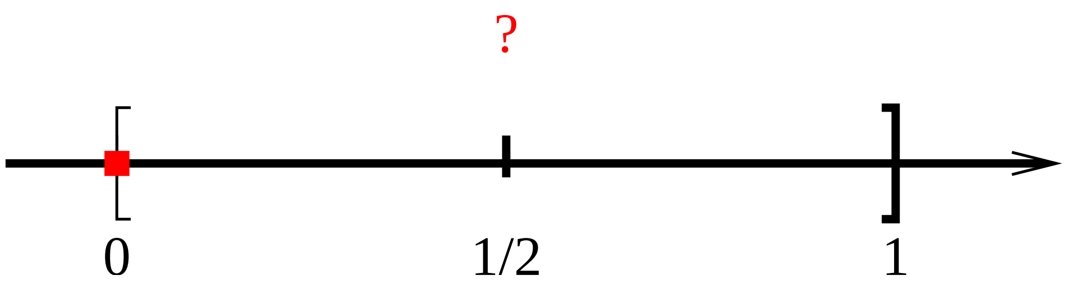
- *acceptable* : predicate over $\mathbb{N}^*$
  - ε is acceptable
  - $\sigma \bullet 0$ is acceptable iff σ is acceptable
  - $\sigma \bullet (n+1)$ is acceptable iff $f(\sigma)_g \subseteq \bigcup_{i<n+1}]\alpha_i, \beta_i[$

- $f : n \in \mathbb{N} \mapsto [d_1, d_2]$ with dyadic bounds
  - $f(\varepsilon) := [0,1]$
  - $f(\sigma \bullet 0) := f(\sigma)_g$
  - $f(\sigma \bullet (n+1)) := \begin{cases} f(\sigma)_d \text{ if } \sigma \bullet (n+1) \text{ is acceptable} \\ f(\sigma)_g \text{ otherwise} \end{cases}$

How does the proof work?

Current state of the proof

Known part of A

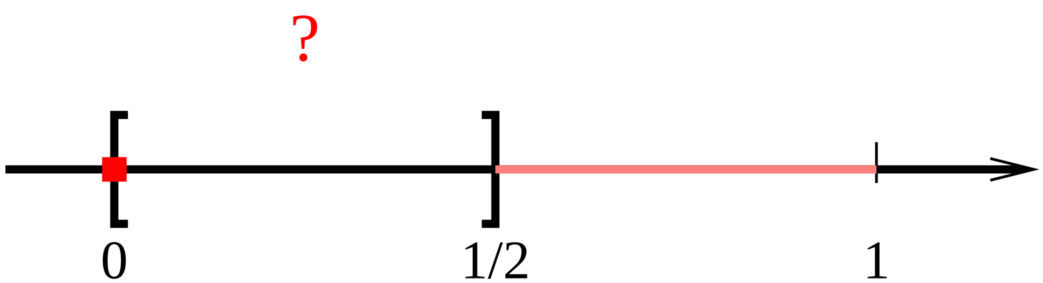


$g(0) =]\alpha_0, \beta_0[$

How does the proof work?

Current state of the proof

Known part of A



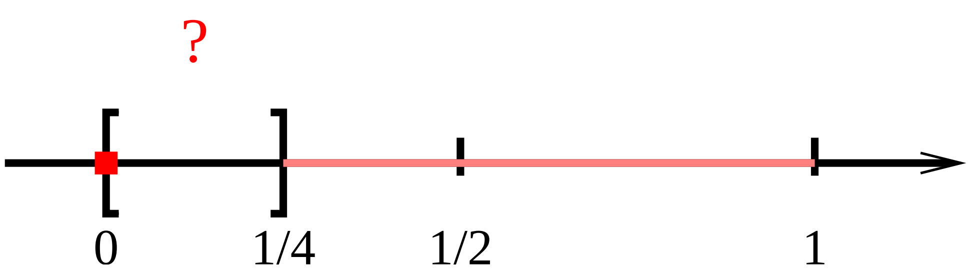
g(0) U g(1)

]α0, β0 [U]α1,β1[

How does the proof work?

Current state of the proof

Known part of A



g(0) U g(1) U g(2)

]α0, β0 [U]α1, β1[U]α2, β2[

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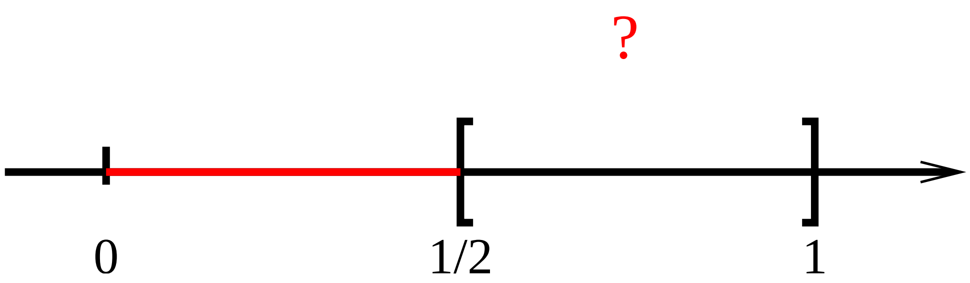


g(0) U g(1) U g(2)

]α0, β0 [U]α1, β1[U]α2, β2[

Roles played by *f* and *acceptable*

- *acceptable* builds certificates that we succeeded in including the closed interval piecewise
- *f* performs the encoding and gives the «current» closed interval

Key lemma of the proof

For each infinite sequence of natural numbers $n_1 n_2 \dots n_l \dots$:

$$\exists k \text{ so that } f(n_1 \dots n_k) \subseteq \cup_{i \leq k}]\alpha_i, \beta_i[$$

Ingredients of the proof:

- density of $\mathbb{Q}$ in $\mathbb{R}$
- completeness of $\mathbb{R}$
- A is *inductive* $(*)$

Predicates over trees

One defines two predicates over $\mathbb{N}^*$:

- $X(\sigma) := f(\sigma)$ is included in $\cup_{i \leq |\sigma|}]\alpha_i, \beta_i[$

- $Y(\sigma) := [\sigma$ acceptable $\Rightarrow f(\sigma)$ is included in a finite union of intervals from $f(\mathbb{N})]$

Bar induction at work

One can show that :

- X is monotonous
- X is included in Y
- Y is hereditary

Bar induction principle yields to $Y(\varepsilon)$:

ε acceptable $\Rightarrow f(\varepsilon)$ is included in a finite union of intervals from $f(\mathbb{N})$

End of the proof

As ε is acceptable,

- $f(\varepsilon) = [0, 1]$ is included in a finite union of $]\alpha_i, \beta_i[$
- hence $[0, 1]$ is included in A

Computational content of bar induction

$$\frac{\begin{array}{ll} \forall \overline{u} \in \mathbb{N}^* \ [X(\overline{u}) \Rightarrow Y(\overline{u})] & \textit{incl} \ (1) \\ \forall \overline{u} \in \mathbb{N}^* \ \forall a \in \mathbb{N} \ [X(\overline{u}) \Rightarrow X(\overline{u} \bullet a)] & \textit{mono} \ (2) \\ \forall \overline{u} \in \mathbb{N}^* \ [\forall a \in \mathbb{N} \ Y(\overline{u} \bullet a)] \Rightarrow Y(\overline{u}) & \textit{here} \ (3) \\ \forall \sigma \in \mathbb{N}^\omega \ \exists k \mid X(\sigma_1 \ldots \sigma_k) & \textit{stop} \ (4) \end{array}}{\forall \overline{u} \ Y(\overline{u})}$$

Realizing hypothesis

- *incl* term of type (1):
  proof of $\overline{u} \in X$ $\rightarrow$ proof of $\overline{u} \in Y$
- *mono* term of type (2):
  $\overline{u} \in \mathbb{N}^* \rightarrow a \in \mathbb{N} \rightarrow$proof of $X(\overline{u}) \rightarrow$ proof of $X(\overline{u} \bullet a)$
- *here* term of type (3):
  $\overline{u} \in \mathbb{N}^* \rightarrow [a \in \mathbb{N} \rightarrow$ proof of $Y(\overline{u} \bullet a)] \rightarrow Y(\overline{u})$
- *stop* [*no need for the computation*]

Realizing the axiom

One can realize the axiom with the following term:

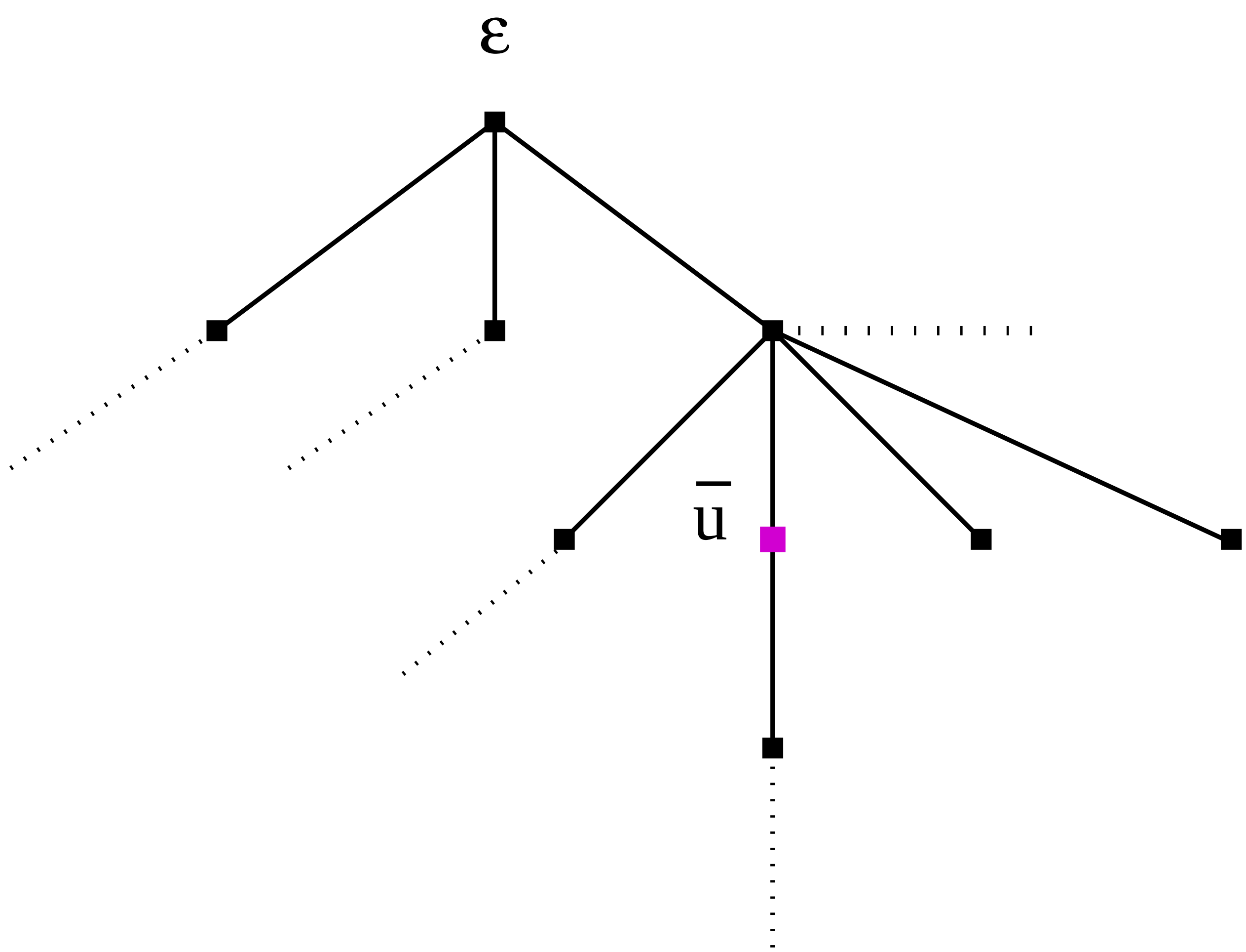
$BI\_rec\ X\ Y\ incl\ mono\ here\ \overline{u}\ :=$
Cases $(X\_dec\ \overline{u})$ of
$(Inx) => (incl\ \overline{u})$
$|(Out\ \_) =>$
$[here\ \overline{u}\ \lambda a.(BI\_rec\ X\ Y\ stop\ incl\ mono\ here\ (\overline{u} \bullet a))]$
End.

Can we realize in fact?

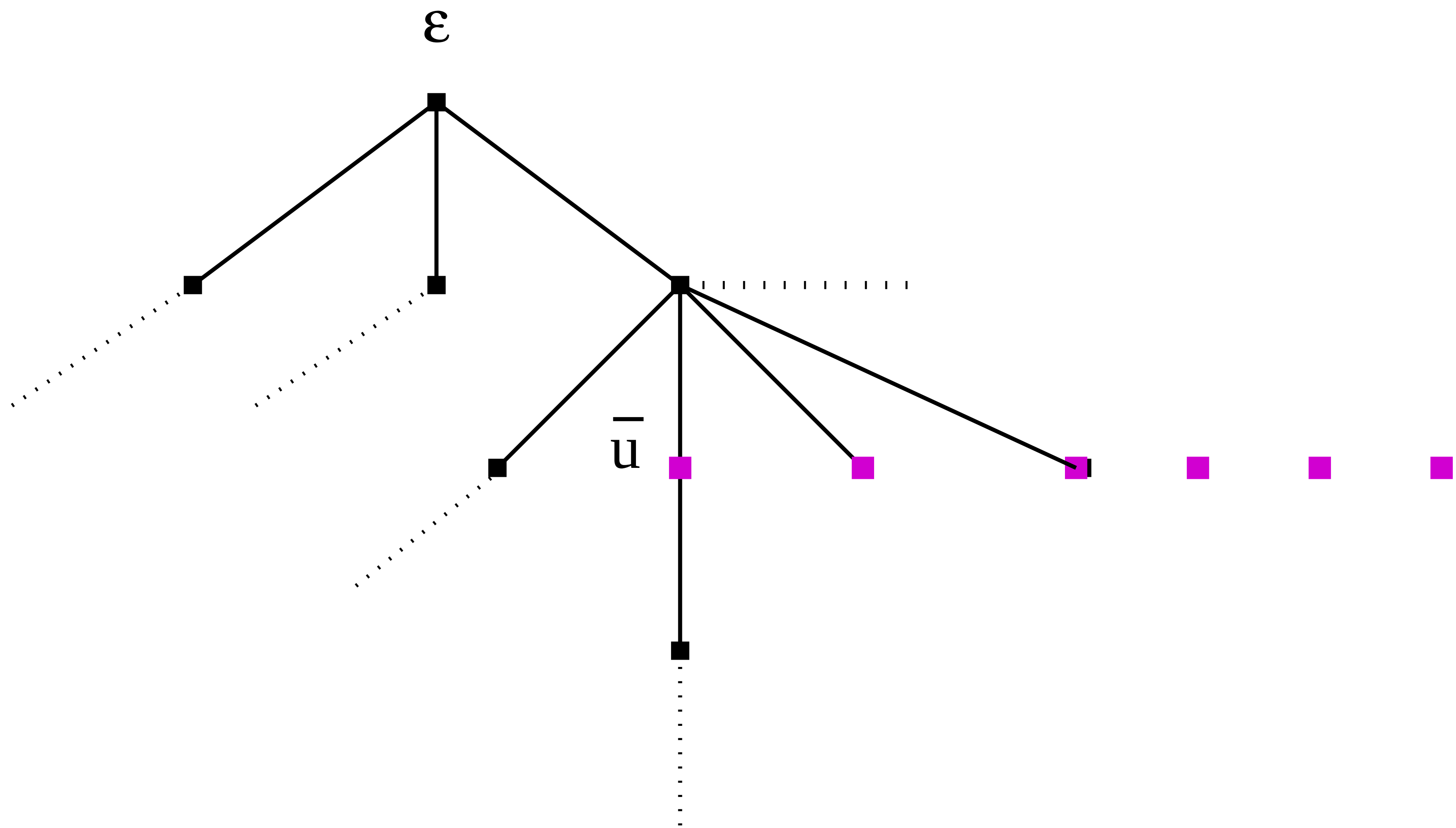
- We get a finite representation of a non terminating term
- Is the proof itself associated with an infinite process ?

$\hookrightarrow$ Let’s examine the way bar induction is used in the proof

«Almost» finite trees



«Almost» finite trees



Computational content of bar induction

Follow the way in the tree which constructs the proof

$=$

Enumerate the intervals $]\alpha_i, \beta_i[$ until $[0, 1]$ is covered

$\hookrightarrow$ Proof of termination of this quite trivial algorithm

Main features of the proof

- Works directly with the real numbers
- Does not rely on a particular construction of $\mathbb{R}$

Position of the OI principle?

Two (related) questions :

- Is it the right axiom to take ?
- Position in the axiomatization of $\mathbb{R}$?

...remain to be investigated

Hierarchy of statements