Tutorial Formalization of Algebraic Topology *Talk 1*

The mathematics to formalize

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Summary

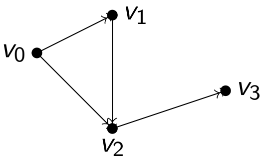
- Plan of the tutorial.
- Introduction to the first talk.
- Simplicial sets.
- The category Δ^*.
- Chain complexes.
- Reductions.
- Basic Perturbation Lemma.
- Effective Homology.
- Bicomplexes.

Plan of the tutorial

- *Talk 1:*
  The mathematics to formalize.
  1. Simplicial Topology.
  2. Basic Perturbation Lemma.
  3. Effective Homology and Bicomplexes.
- *Talk 2:* (from 1.2)
  Isabelle/HOL: First proving, then extracting code.
  (Joint work with J. Aransay and C. Ballarin)
- *Talk 3:* (from 1.3)
  Coq: Algebraic structures, effective homology and type theory.
  (Joint work with C. Domínguez)
- *Talk 4:* (from 1.1)
  ACL2: Going down to first order. The case of Simplicial Topology.
  (Joint work with L. Lambán, F.J. Martín-Mateos and J.L. Ruiz-Reina)

Introduction

A (directed) graph:



Abstractly:

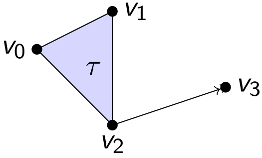
- $V = \{v_0, v_1, v_2, v_3\}, E = \{e_0, e_1, e_2, e_3\}$
- $e_0 = (v_0, v_1), e_1 = (v_0, v_2) \ldots$
- That is to say: $E \subseteq V \times V$.

Other combinatorial description:

- $\partial_0 \colon E \rightarrow V, \partial_1 \colon E \rightarrow V$
- $\partial_0(e_0) = \partial_0(v_0, v_1) := v_1, \partial_0(e_1) = \partial_0(v_0, v_2) := v_2$
- $\partial_1(e_0) = \partial_0(v_0, v_1) := v_0, \partial_1(e_1) = \partial_0(v_0, v_2) := v_0 \ldots$
- $\partial_0 =$ target, $\partial_1 =$ source

Introduction

A (triangulated) space K:



Abstractly:

- Any (ordered) subset of (v_0, v_1, v_2) and (v_2, v_3) is in K.

Other combinatorial description:

- $\partial_0^{(2)}\colon K_2 \to K_1, \partial_1^{(2)}\colon K_2 \to K_1, \partial_2^{(2)}\colon K_2 \to K_1$
- $\partial_0^{(1)}\colon K_1 \to K_0, \partial_1^{(1)}\colon K_1 \to K_0$
- $\partial_0^{(2)}(v_0, v_1, v_2) := (v_1, v_2), \ldots$
- But now it is needed that: $\partial_0^{(1)}\partial_0^{(2)}(\tau) = \partial_0^{(1)}\partial_1^{(2)}(\tau)$, ...

Simplicial Complexes

Given an ordered set V, a *simplicial complex* K is a subset of

$$OrderedList(V) = \{(v_0, v_1, \ldots, v_m)\colon v_0 < v_1 < \ldots < v_m\},$$

such that any ordered sublist of an element of K is again in K.
The dimension of a list $(v_0, v_1, \ldots, v_m)$ is m. Thus K is naturally graded by the dimension of its *simplexes*.
A simplicial complex K admits another combinatorial description:

- $\partial_i^{(n)}\colon K_n \to K_{n-1}, 0 \leq i \leq n$
- satisfying: $\partial_i^{(n)}\partial_j^{(n+1)} = \partial_j^{(n)}\partial_{i+1}^{(n+1)}$, if $0 \leq j \leq i \leq n$.
- ($\partial_i =$ erasing the i-th element in a list)

Theorem

Let K be a subset of $OrderedList(V)$. K is a simplicial complex if and only if the operators $\{\partial_i^{(n)}\}$ are closed on K.

Simplicial Complexes with degeneracies

If we allow the lists to have duplicates, that is to say if we consider as simplexes elements of $\{(v_0, v_1, \ldots, v_m) \colon v_0 \leq v_1 \leq \ldots \leq v_m\}$, we can define new operators η_i which repeat the i-th element of a list.
Then, the following identities hold:

$$\partial_i^{(n)}\partial_j^{(n+1)} = \partial_j^{(n)}\partial_{i+1}^{(n+1)} \qquad \text{if } 0 \leq j \leq i \leq n \qquad (1)$$

$$\eta_i^{(n+1)}\eta_j^{(n)} = \eta_{j+1}^{(n+1)}\eta_i^{(n)} \qquad \text{if } 0 \leq i \leq j \leq n \qquad (2)$$

$$\partial_i^{(n+1)}\eta_j^{(n)} = \eta_{j-1}^{(n-1)}\partial_i^{(n)} \qquad \text{if } 0 \leq i < j \leq n \qquad (3)$$

$$\partial_i^{(n+1)}\eta_j^{(n)} = id \qquad \text{if } 0 \leq i = j \leq n \qquad (4)$$

$$\text{or } 0 < i = j+1 \leq n+1 \qquad (5)$$

$$\partial_i^{(n+1)}\eta_j^{(n)} = \eta_j^{(n-1)}\partial_{i-1}^{(n)} \qquad \text{if } 0 < j+1 < i < n \qquad (6)$$

Simplicial Sets

If we abstract from the previous definition, we can define a *simplicial set* K as a graded set $\{K_n\}_{n\in\mathbb{N}}$ endowed with operations $\partial_i^{(n)}\colon K_n \to K_{n-1}$ and $\eta_i^{(n)}\colon K_n \to K_{n+1}$, $\forall 0 \le i \le n \in \mathbb{N}$ satisfying the *simplicial identities*:

$\partial_i^{(n)}\partial_j^{(n+1)} = \partial_j^{(n)}\partial_{i+1}^{(n+1)}$	if $0 \le j \le i \le n$	(1)
$\eta_i^{(n+1)}\eta_j^{(n)} = \eta_{j+1}^{(n+1)}\eta_i^{(n)}$	if $0 \le i \le j \le n$	(2)
$\partial_i^{(n+1)}\eta_j^{(n)} = \eta_{j-1}^{(n-1)}\partial_i^{(n)}$	if $0 \le i < j \le n$	(3)
$\partial_i^{(n+1)}\eta_j^{(n)} = id$	if $0 \le i = j \le n$	(4)
	or $0 < i = j+1 \le n+1$	(5)
$\partial_i^{(n+1)}\eta_j^{(n)} = \eta_j^{(n-1)}\partial_{i-1}^{(n)}$	if $0 < j+1 < i < n$	(6)

The category Δ^*

- Objects: $\mathbf{n} = \{0, 1, \ldots, n\}, \forall n \in \mathbb{N}$.
- Morphisms: $\mu : \mathbf{n} \to \mathbf{m}$, increasing.
- Each morphism μ can be written as $\mu_{mono} \circ \mu_{epi}$
- Distinguished morphisms:
  - (Mono) $\{\delta_i^{(n)} : \mathbf{n} \to \mathbf{n+1} \; ; \; 0 \leq i \leq n \}$,
    with $\delta_i^{(n)}(j) = j$ if $j < i$ and $\delta_i^{(n)}(j) = j + 1$ if $j \geq i$.
  - (Epi) $\{\sigma_i^{(n)} : \mathbf{n} \to \mathbf{n\text{-}1} \; ; \; 0 \leq i \leq n - 1 \}$
    with $\sigma_i^{(n)}(j) = j$ if $j \leq i$ and $\sigma_i^{(n)}(j) = j - 1$ if $j > i$.
- Each morphism μ can be written in a unique way as:

$$\mu = \delta_{j_s} \ldots \delta_{j_1} \sigma_{i_t} \ldots \sigma_{i_1}, \text{ with } 0 \leq i_t < \ldots < i_1 \text{ and } 0 \leq j_1 < \ldots < j_s.$$

(Important remark: superindices skipped)

Identities in Δ^*

The morphisms δ_i and σ_i satisfy a series of identities:

$$\delta_j\delta_i = \delta_{i+1}\delta_j \qquad \text{if } i \geq j \tag{1}$$

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1} \qquad \text{if } i \leq j \tag{2}$$

$$\sigma_j\delta_i = \delta_i\sigma_{j-1} \qquad \text{if } i < j \tag{3}$$

$$\sigma_j\delta_i = id \qquad \text{if } i = j \tag{4}$$

$$\qquad \text{or } i = j+1 \tag{5}$$

$$\sigma_j\delta_i = \delta_{i-1}\sigma_j \qquad \text{if } i > j+1 \tag{6}$$

What?

Compare:

$$
\begin{aligned}
\delta_j\delta_i &= \delta_{i+1}\delta_j && \text{if } i \geq j && (1)\\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1} && \text{if } i \leq j && (2)\\
\sigma_j\delta_i &= \delta_i\sigma_{j-1} && \text{if } i < j && (3)\\
\sigma_j\delta_i &= id && \text{if } i = j && (4)\\
& && \text{or } i = j+1 && (5)\\
\sigma_j\delta_i &= \delta_{i-1}\sigma_j && \text{if } i > j+1 && (6)
\end{aligned}
$$

$$
\begin{aligned}
\partial_i\partial_j &= \partial_j\partial_{i+1} && \text{if } i \geq j && (1)\\
\eta_i\eta_j &= \eta_{j+1}\eta_i && \text{if } i \leq j && (2)\\
\partial_i\eta_j &= \eta_{j-1}\partial_i && \text{if } i < j && (3)\\
\partial_i\eta_j &= id && \text{if } i = j && (4)\\
& && \text{or } i = j+1 && (5)\\
\partial_i\eta_j &= \eta_j\partial_{i-1} && \text{if } i > j+1 && (6)
\end{aligned}
$$

Another definition of Simplicial Set

A *simplicial set* is a (contravariant) functor $K : \Delta^* \rightarrow Set$.

- $K_n := K(\mathbf{n})$ (n-simplexes)
- $\partial_i := K(\delta_i)$ (faces)
- $\eta_i := K(\sigma_i)$ (degeneracies)

Theorem

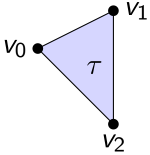
Given a simplicial set K and a simplex $x \in K_n$, there exists a unique expression $x = \eta_{i_1} \ldots \eta_{i_t} \overline{x}$, with $\overline{x}$ non-degenerate (i.e. $\overline{x} \notin Im(\eta_j), \forall j$), and $0 \leq i_t < \ldots < i_1$ (t could be equal to 0).

Recall, in Δ^*:

$\mu = \delta_{j_s} \ldots \delta_{j_1} \sigma_{i_t} \ldots \sigma_{i_1}$, with $0 \leq i_t < \ldots < i_1$ and $0 \leq j_1 < \ldots < j_s$.

Simplicial Sets and Chain Complexes

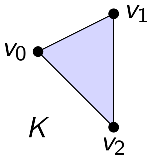
- Let K be a simplicial set.
- Define: $C_n(K) := \mathbb{Z}[K_n]$, free $\mathbb{Z}$-module generated by n-simplexes.
- Define: $d_n(x) := \sum_{i=0}^{n}(-1)^i \partial_i x$ over generators, and extend linearly.
- Then: $d_n \circ d_{n+1} = 0$.

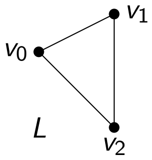


$$d_1 d_2(\tau) = d_1(\partial_0(v_0, v_1, v_2) - \partial_1(v_0, v_1, v_2) + \partial_2(v_0, v_1, v_2)) = \ldots = 0$$

Homology groups

- $d_n \circ d_{n+1} = 0 \implies Im(d_{n+1}) \subseteq Ker(d_n) \subseteq C_n(K)$
- We can define
  $H_n(K) := Ker(d_n)/Im(d_{n+1})$, the n-th *homology group* of K.
- Geometrical meaning:





- $c := (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$ defines a *cycle* ($\in Ker(d_1)$), both in K and L.
- $H_1(K) = 0$, but $H_1(L) = \mathbb{Z}$ generated by c, since it is not a *boundary* ($\notin Im(d_2)$).

Degenerate and non-degenerate simplexes

- A simplex $x \in K_n$ is *degenerate* if $x = \eta_i(\bar{x})$ for some $\bar{x} \in K_{n-1}$ and some i with $0 \leq i < n$.
- Otherwise: x is called *non-degenerate*.
- In the simplicial *complex* case: non-degenerate = without duplicates.
- Let us call K_n^{ND} the set of non-degenerate n-simplexes of a simplicial set K.
- And let us call K_n^{D} the set of degenerate n-simplexes.

Different chain complexes, equal homology groups

- We define a new chain complex with $D_n(K) := \mathbb{Z}[K_n^D]$ and as differential the restriction over $D(K)$ of that of $C(K)$.
- *Remark*: $d|_{D(K)}$ is well defined.
- Define $C^{ND}(K) := C(K)/D(K)$.
- On the contrary, if $\overline{C_n(K)} := \mathbb{Z}[K_n^{ND}]$, the differential is not well-defined
- ...but can be slightly modified to produce another chain complex associated with K: $\overline{C_n(K)}$.
- $C^{ND}(K)$ and $\overline{C(K)}$ are isomorphic
- ...and thus it is the same for $H(C^{ND}(K))$ and $H(\overline{C(K)})$.
- What about the relation between $H(K)$ and $H(C^{ND}(K))$?

General chain complexes

- A chain complex is $\{C_n, d_n)\}_{n\in\mathbb{Z}}$, where each C_n is an abelian group, and each $d_n : C_n \to C_{n-1}$ is a homomorphism satisfying $d_{n+1} \circ d_n = 0, \forall n \in \mathbb{Z}$.
- *Examples:* Chain complexes associated with simplicial sets (here $C_n = 0, \forall n < 0$; it is called a *positive* chain complex).
- *Homology groups:* $H_n(C, d) := Ker(d_n)/Im(d_{n+1})$.
- Given two chain complexes $\{C_n, d_n)\}_{n\in\mathbb{Z}}$ and $\{C'_n, d'_n)\}_{n\in\mathbb{Z}}$, a *chain morphism* between them is a family f of group homomorphisms $f_n : C_n \to C'_n, \forall n \in \mathbb{Z}$ satisfying $d'_n \circ f_n = f_{n-1} \circ d_n, \forall n \in \mathbb{Z}$.

Reductions

- Given two chain complexes $C := \{C_n, d_n)\}_{n\in\mathbb{Z}}$ and $C' := \{C'_n, d'_n)\}_{n\in\mathbb{Z}}$ a *reduction* between them is (f, g, h) where
  - $f : C \to C'$ and $g : C' \to C$ are chain morphisms
  - and h is a family of homomorphisms (called *homotopy operator*) $h_n : C_n \to C_{n+1}$.

  satisfying

  1. $f \circ g = 1$
  2. $d \circ h + h \circ d + g \circ f = 1$
  3. $f \circ h = 0$
  4. $h \circ g = 0$
  5. $h \circ h = 0$

- If $(f, g, h) : C \to C'$ is a reduction, then $H(C) \cong H(C')$.
- Let K be a simplicial set, then there exists a reduction $(f, g, h) : C(K) \to C^{ND}(K)$.

Basic Perturbation Lemma

- Given a chain complex (C, d), a *perturbation* for it is a family ρ of group homomorphisms $\rho_n : C_n \rightarrow C_{n-1}$ such that $(C, d + \rho)$ is again a chain complex (that is to say: $(d + \rho) \circ (d + \rho) = 0$).
- A reduction $(f, g, h) : (C, d) \rightarrow (C', d')$ and a perturbation ρ for (C, d) are *locally nilpotent* if $\forall x \in C_n, \exists m \in \mathbb{N}$ such that $(h \circ \rho)^m(x) = 0$.

Basic Perturbation Lemma

Let $(f, g, h) : (C, d) \rightarrow (C', d')$ be a reduction and be ρ a perturbation for (C, d) which are locally nilpotent. Then there exists a reduction $(f_\infty, g_\infty, h_\infty) : (C, d + \rho) \rightarrow (C', d'_\infty)$.

Sergeraert's effective homology

- From now on, all the groups in chain complexes will be *free* abelian groups with an explicit basis.
- That is: $C_n = \mathbb{Z}[B_n]$. (Example: $C_n(K) = \mathbb{Z}[K_n]$.)
- A chain complex is *effective*, if $\forall n \in \mathbb{Z}, B_n$ is a finite set presented as a list of elements.
- On the contrary, a chain complex is called *locally effective* if the only known data on their bases are their characteristic functions and an equality test.
- A chain complex with *(strong) effective homology* is a data structure $[C, E, f, g, h]$ where C is a chain complex (possibly locally effective), E is an *effective* chain complex, and $(f, g, h) : C \to E$ is a reduction.

Basic Perturbation Lemma Algorithm

Given a chain complex (C, d) with effective homology and ρ a perturbation for it satisfying the local nilpotency condition, then $(C, d + \rho)$ is a chain complex with effective homology.

Bicomplexes

- A (first quadrant) *bicomplex* C is a family of pairs $(C_{p,*}, f_p)_{p\in\mathbb{N}}$ with $(C_{p,*})_{p\in\mathbb{N}}$ a family of *positive* chain complexes and $(f_p\colon C_{p+1,*} \to C_{p,*})_{p\in\mathbb{N}}$ a family of chain morphisms, such that $f_p \circ f_{p+1} = 0$.
- Given a bicomplex $C = \{C_{p,q}, d_{p,q}, f_{p,q}\}_{p,q\in\mathbb{N}}$, the *totalization* of C is the chain complex $T(C) = (T(C)_n, d_n)_{n\in\mathbb{N}}$ where $T(C)_n = \oplus_{p+q=n} C_{p,q}$ and $d_n = \oplus_{p+q=n}(d_{p,q} \oplus (-1)^p f_{p,q})$.

Effective homology of bicomplexes

Let C be a bicomplex $(C_{p,*}, f_p)_{p\in\mathbb{N}}$ such that each chain complex $C_{p,*}$ is with effective homology. Then the total chain complex $T(C) = (T(C)_n, d_n)_{n\in\mathbb{N}}$ is with effective homology.

Two proofs:

- By using the Basic Perturbation Lemma.
- As an iteration of *mapping cones*.