Unique Solutions

Attempts to Demystify a Mystery

Peter Schuster

Pure Mathematics, University of Leeds

Work in progress started during a Feodor Lynen Research Fellowship of the Alexander-von-Humboldt Foundation

Framework

Bishop-style constructive mathematics without countable choice.

Intuitionistic logic. Suitable fragment of CZF. Unique Choice.
No non-unique countable choice, let alone dependent choice.
Completions without sequences; real numbers: Dedekind cuts.

Continuity: uniform continuity on every compact domain.
Compactness: total boundedness plus completeness.

Setting

Let S be a metric space and $F : S \to \mathbb{R}$ a continuous function.

If S is totally bounded and F uniformly continuous, then $\inf F$ can be computed, in which case it may be assumed that $\inf F = 0$.

If S is compact, can one locate a minimum of F? That is, find a point of S at which F attains its infimum, or simply a root of F?

Heuristics: constructive solutions are continuous in the parameters. Hence uniqueness of the solution is needed to rule out discontinuity.

Variants of uniqueness

Let $F \geqslant 0$; denote points of S by y, y' and real numbers > 0 by ε, δ.

Any such F has *uniformly at most one* root if

$$\forall \delta\, \exists \varepsilon\, \forall y, y' \left[F(y) < \varepsilon \wedge F(y') < \varepsilon \Rightarrow d(y, y') < \delta \right]$$

or, equivalently,

$$\forall \delta\, \underline{\exists \varepsilon\, \forall y, y'} \left[d(y, y') \geqslant \delta \Rightarrow F(y) \geqslant \varepsilon \vee F(y') \geqslant \varepsilon \right] .$$

In this case F has *at most one* root: i.e.,

$$\forall \delta\, \underline{\forall y, y'\, \exists \varepsilon} \left[d(y, y') \geqslant \delta \Rightarrow F(y) \geqslant \varepsilon \vee F(y') \geqslant \varepsilon \right]$$

or more simply but equivalently

$$\forall y, y' \left[y \neq y' \Rightarrow F(y) > 0 \vee F(y') > 0 \right] .$$

Theorem 1 *Let* S *be a complete metric space and* $F : S \to \mathbb{R}$ *a uniformly continuous function. If* $\inf F = 0$ *and* F *has uniformly at most one root, then there is* $y \in S$ *with* $F(y) = 0$.

This well known metatheorem has a considerable history:

Lifshitz 1971, Gelfond 1972, Kreinovich 1979, Bridges 1980, Aczel 1987, Ko 1986, Kohlenbach 1993, Weihrauch 2000, Oliva 2002, Kohlenbach-Oliva 2003, Bauer-Taylor 2005, Brattka 2008, . . .

(to mention for each author only the first printed occurrence)

The metatheorem

— can be traced back to Russian recursive mathematics;

— has proved productive in constructive/computable analysis;

— stood right at the beginnings of the so-called proof mining.

The uniqueness hypothesis helps to find the root above any “pure existence proof” tied together with the use of classical logic.

Two (semi-)classical short cuts

If S has the Bolzano-Weierstraß property, then no uniqueness hypothesis at all is necessary.

If S has the Heine-Borel property, then the non-uniform uniqueness precondition suffices.

The Weak König Lemma is equivalent to (Ishihara 1990):

Every continuous function on a compact space attains its infimum.

Brouwer's Fan Theorem is equivalent to (J. Berger, Bridges, Sch. 2005):

Every continuous function on a compact space that has at most one minimum attains its infimum.

Uniqueness with parameters

Let S, T be metric spaces and $F : T \times S \to \mathbb{R}$ with $F \geqslant 0$ such that

$$\forall\varepsilon\underline{\forall x\exists\delta}\forall x'\forall y,y'\left[d(x,x')<\delta\wedge F(x,y)<\delta\wedge F(x',y')<\delta\Rightarrow d(y,y')<\varepsilon\right],$$

or even

$$\forall\varepsilon\underline{\exists\delta\forall x}\forall x'\forall y,y'\left[d(x,x')<\delta\wedge F(x,y)<\delta\wedge F(x',y')<\delta\Rightarrow d(y,y')<\varepsilon\right].$$

Before any talk of existence, note that $F(x,y)=0$ defines a pointwise (or even uniformly) continuous partial function $x \mapsto y$.

(Look at the case $F(x,y)=0$ and $F(x',y')=0$, first with $x=x'$.)

In other words: Uniqueness with parameters implies continuity.

A parametrised version of the metatheorem is known equally well:

Corollary 2 *Let $F : S \times T \to \mathbb{R}$ be as above. If, in addition, S is complete, and $F(x, \cdot)$ uniformly continuous with* $\inf F(x, \cdot) = 0$ *for each $x \in T$, then there is a pointwise or even uniformly continuous $f : T \to S$ with $F(x, f(x)) = 0$ for all $x \in T$.*

The case $x = x'$ already of the non-uniform hypothesis

$$\forall \varepsilon \forall x \exists \delta \forall x' \forall y, y' \left[d(x, x') < \delta \wedge F(x, y) < \delta \wedge F(x', y') < \delta \Rightarrow d(y, y') < \varepsilon \right]$$

says that $F(x, \cdot)$ has uniformly at most one root for every $x \in T$; whence the partial function $x \mapsto y$ defined by $F(x, y) = 0$ is total.

This subsumes the Implicit Functions Theorem (Diener-Sch. 2009).

Proving the metatheorem with countable choice (folklore)

Since $\inf F = 0$ one can choose (!) a sequence (y_n) in S with $F(y_n) < 1/n$, which is a Cauchy sequence because F has uniformly at most one root. Hence if S is complete, then (y_n) has a limit y in S, for which $F(y) = 0$ by the continuity of F.

One only needs that S is complete, and F sequentially continuous. Uniform uniqueness, however, is essential to get a *Cauchy* sequence.

Even if S fails to be complete, the given data are converted—by countable choice—into an element of the completion of S.

The problem thus provides us, in a sense, with its own solution.

. . . and without countable choice (Sch. 2009)

The completion of S is the set $\widehat{S}$ of locations (Richman 2000).

Similar methods to define completions without sequences:
Mulvey 1979, Burden and Mulvey 1979, Stolzenberg 1988,
Vickers 2005, Fox 2005, Palmgren 2007, . . .

Let $\mathbb{R}$ denote the set of Dedekind reals: that is, located cuts in $\mathbb{Q}$.

A *location* on S is a function $f : S \to \mathbb{R}$ with $\inf f = 0$ and

$$|f(y) - f(z)| \leqslant d(y, z) \leqslant f(y) + f(z) .$$

The set $\widehat{S}$ of all locations on S is a metric space with metric

$$d(f, g) = \sup |f - g| = \inf (f + g) .$$

There is the isometric embedding

$$S \hookrightarrow \widehat{S}\,, \;\; z \mapsto \widehat{z} = d\,(z,\,\cdot\,)\;,$$

along which (each point of) S is identified with its image in $\widehat{S}$.

As usual, S is dense in $\widehat{S}$, and S is *complete* if S equals $\widehat{S}$:
that is, for every $f \in \widehat{S}$ there is $z \in S$ with $f = \widehat{z}$.

Needless to say, $\widehat{S}$ is complete; and so is $\mathbb{R}$ for $\mathbb{R} \cong \widehat{\mathbb{Q}}$.

Every location measures the distance between itself and the points:

$$d\,(f, \widehat{z}) = f\,(z)\;.$$

Every location f on S is uniformly continuous with $\inf f = 0$, has uniformly at most one root, and satisfies

$$f(z) = \lim_{f(y)\to 0} d(z,y) \; .$$

More generally, if $F : S \to \mathbb{R}$ with $\inf F = 0$ has uniformly at most one root, then the corresponding limit exists for every $z \in S$:

$$\lim_{F(y)\to 0} d(z,y) \; .$$

Lemma 3 *If* $F : S \to \mathbb{R}$ *with* $\inf F = 0$ *is uniformly continuous and has uniformly at most one root, then*

$$f_F(z) = \lim_{F(y)\to 0} d(z,y)$$

defines $f_F \in \widehat{S}$ *with* $\widehat{F}(f_F) = 0$ *.*

Even more clearly, the problem provides us with its own solution!

Note that every $\varphi : S \to T$ that is uniformly continuous on bounded subsets extends uniquely to a mapping $\widehat{\varphi} : \widehat{S} \to \widehat{T}$ with

$$\widehat{\varphi}(f)(z) = \lim_{f(y) \to 0} d(\varphi(y), z)$$

for every $z \in T$ which is uniformly continuous on bounded subsets. If S and T are complete, then $\widehat{\varphi} = \varphi$.

Example If $S = \mathbb{R} \setminus \{a\}$ and $F(t) = |t - a|^k$ with $k \geqslant 1$, then

$$f_F(t) = \lim_{F(s) \to 0} d(t, s) = \lim_{s \to a} d(t, s) = d(t, a) = |t - a| .$$

In particular, $f_F = F$ precisely when $k = 1$, which is the only case in which F is a location.

But why does uniform uniqueness help to find the solution at all?

The unique solution property: an equivalent of completeness

Definition A metric space has the *unique solution property* if for every uniformly continuous $F : S \to \mathbb{R}$ with $\inf F = 0$ which has uniformly at most one root there is $y \in S$ with $F(y) = 0$.

The metatheorem thus says that every complete metric space has the unique solution property. The converse, however, is also valid:

Theorem 4 *A metric space has the unique solution property* if and only if *it is complete.*

In fact, S is complete already when every location on S has a root in S: if $f \in \widehat{S}$ has the root $y \in S$, then $f = \widehat{y}$ because $d(f, \widehat{y}) = f(y)$.

Alternative proof, for completions with Cauchy sequences:

If (y_n) is a Cauchy sequence in S, then

$$f(y) = \lim_{n\to\infty} d(y, y_n)$$

defines a location f on S such that

$$f(y) = 0 \Leftrightarrow \lim_{n\to\infty} y_n = y \,.$$

Moduli of convergence and of uniqueness correspond to each other.

"Cauchy sequence" and uniform uniqueness have the same form:

$$\forall \delta \, \exists N \, \forall k, k' \, \left[k \geqslant N \wedge k' \geqslant N \;\Rightarrow\; d(y_k, y_{k'}) < \delta \right]$$

$$\forall \delta \, \exists \varepsilon \, \forall y, y' \, \left[F(y) < \varepsilon \wedge F(y') < \varepsilon \;\Rightarrow\; d(y, y') < \delta \right]$$

The Banach Fixed Point Theorem: a simple-minded example

Let $h : S \to S$ be such that there is $\lambda < 1$ with

$$d(h(y), h(y')) \;\leqslant\; \lambda \cdot d(y, y')\,.$$

First, h has approximate fixed points: for any $y_0 \in S$,

$$d(h^k(y_0), h(h^k(y_0)) \;\leqslant\; \lambda^k \cdot d(y_0, h(y_0))\,.$$

Next, h has uniformly at most one fixed point:

$$\begin{aligned} d(y, y') \;&\leqslant\; d(h(y), h(y')) + \Big[d(y, h(y)) + d(y', h(y')\Big] \\ \Rightarrow\;\; (1-\lambda)\cdot d(y, y') \;&\leqslant\; d(y, h(y)) + d(y', h(y'))\,. \end{aligned}$$

Along the same lines two typical applications, the Implicit Functions Theorem and the Picard-Lindelöf Theorem, can be settled directly, without any need to invoke the Banach Fixed Point Theorem.

Incomplete Bibliography

Berger, J., and H. Ishihara, Brouwer's fan theorem and unique existence in constructive analysis. *Math. Log. Quart.* 51 (2005), 369–373

Schwichtenberg, H., A direct proof of the equivalence between Brouwer's fan theorem and König's lemma with a uniqueness hypothesis. *J. UCS* 11 (2005), 2086–2095

Berger, J., D. Bridges, and P. Sch., The fan theorem and unique existence of maxima. *J. Symbolic Logic* 71 (2006), 713–720

P. Sch., Unique solutions. *Math. Log. Quart.* 52 (2006), 534–539. Corrigendum: *Math. Log. Quart.* 53 (2007), 214

Ishihara, H., Unique existence and computability in constructive reverse mathematics. In: S.B. Cooper, B. Löwe, A. Sorbi, eds., *Computation and Logic in the Real World.* CiE 2007. Springer. *Lect. Notes Comput. Sci.* 4497 (2007), 368–377

P. Sch., Problems, solutions, and completions. *J. Logic Algebr. Program.* 79 (2010), 84–91

[a digression on limits]

Let $h : S \to \mathbb{R}$ and $a \in \mathbb{R}$ with $\inf_{x \in S} |h(x) - a| = 0$.

For every $g : S \to \mathbb{R}$ and $b \in \mathbb{R}$, we write $\lim_{h(x) \to a} g(x) = b$ whenever

$$\forall \varepsilon \, \exists \delta \, \forall x \ (|h(x) - a| < \delta \Rightarrow |g(x) - b| < \varepsilon) \ .$$

(If $S = \mathbb{R}$ and $h(x) = x$, this means nothing but $\lim_{x \to a} g(x) = b$.)

The limit—if it exists—is uniquely determined. But when does it exist?

A (necessary and) sufficient condition for the existence of the limit is

$$\forall\varepsilon\,\exists\delta\,\forall x, y\ \left(|h(x) - a| < \delta \wedge |h(y) - a| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon\right).$$

In fact, the subset L of $\mathbb{Q}$ characterised by

$$r \in L \Leftrightarrow \exists s \in \mathbb{Q}\ \left[r < s \wedge \exists\delta\,\forall x\ \left(|h(x) - a| < \delta \Rightarrow s < g(x)\right)\right]$$

is a lower cut in $\mathbb{Q}$ defining $b \in \mathbb{R}$ with $\lim_{h(x)\to a} g(x) = b$.

[end of digression]

Proof of Lemma For every x the limit exists and is uniquely determined by x; whence unique choice suffices to obtain the function f_F. It is routine to verify that f_F is a location on S; we still have to see that $\widehat{F}(f_F) = 0$.

Since $f_F(x) = \lim_{F(y)\to 0} d(x, y)$, we indeed have

$$d\left(\widehat{F}(f_F), 0\right) = \widehat{F}(f_F)(0) = \lim_{f_F(x)\to 0} \overbrace{d(F(x), 0)}^{F(x)} \stackrel{(\dagger)}{=} 0 \ .$$

As for (†), it is straightforward to give a rigorous version of the following argument: if $f_F(x)$ is small, then x is close—by the definition of f_F—to some y with $F(y)$ small, so that $F(x)$ is small. *q.e.d.*

The Fan Theorem and unique existence: a digression

Brouwer's Fan Theorem can equivalently be put as:

FAN *Every decidable binary tree without infinite path is finite.*

This is the contrapositive of the Weak König Lemma:

WKL *Every infinite decidable binary tree has an infinite path.*

With intuitionistic logic, WKL implies FAN (Ishihara 2006).

The logical form of FAN and WKL is

FAN $\forall \alpha \, \exists n \; B(\overline{\alpha} n) \;\Rightarrow\; \exists n \, \forall \alpha \; B(\overline{\alpha} n)$

WKL $\forall n \, \exists \alpha \; T(\overline{\alpha} n) \;\Rightarrow\; \exists \alpha \, \forall n \; T(\overline{\alpha} n)$

where B and T are decidable properties of finite binary sequences that are closed under extension and restriction, respectively, and

$$\overline{\alpha} n = (\alpha(0), \ldots, \alpha(n-1))$$

denote the finite initial segments of infinite binary sequences α.

Think of B and T as of each other's complement in $\{0,1\}^*$.

J. Berger, Bridges, and Sch. (2003): FAN is equivalent to

MIN! *Every continuous function on a compact space that has at most one minimum attains its infimum*

J. Berger and Ishihara (2005) exchanged the Minimum Theorem for the Weak König Lemma (see also Schwichtenberg 2005), Cantor's Intersection Theorem, and a Fixed Point Theorem.

All these equivalents of FAN have the following form:

> *If a problem on a compact space has approximate solutions and at most one solution, then it has an exact solution.*

Yet it was all but clear why FAN occurred in this context.

The Positivity Principle

POS *Every continuous function on a compact space that attains only positive values has a positive infimum*

is the equivalent of FAN that was needed to prove MIN!.

With FAN in the form of POS we now know why it occurred there.

Sch. (2006): FAN *is equivalent to*

UAM *If a continuous function on a compact space has at most one minimum, then it has uniformly at most one minimum.*

This sharpens FAN $\Rightarrow$ MIN!, because UAM $\Rightarrow$ MIN!.

Proof We already know that MIN! $\Rightarrow$ FAN and FAN $\Rightarrow$ POS.

To show the missing link POS $\Rightarrow$ UAM, we assume that S is compact, and that $H : S \to \mathbb{R}$ is continuous with $\inf H = 0$.

We will use Bishop's result that $d(x,y) \geqslant \delta$ defines a compact subset of $S \times S$ for all but countably many values of δ.

The condition "H has at most one minimum" can be put as

$$\forall x, y \; (\exists \delta \, d(x,y) \geqslant \delta \;\Rightarrow\; H(x) + H(y) > 0) \; ,$$

which is equivalent to

$$\forall \delta \, \forall x, y \; (d(x,y) \geqslant \delta \;\Rightarrow\; H(x) + H(y) > 0) \; .$$

By POS, this implies

$$\forall \delta \, \exists \varepsilon \, \forall x, y \; (d(x,y) \geqslant \delta \;\Rightarrow\; H(x) + H(y) \geqslant \varepsilon) \; ,$$

which is equivalent to "H has uniformly at most one minimum".

Problems as their own solutions: a trivial observation

Let $\Phi(S)$ consist of all the $F : S \to \mathbb{R}$ with $\inf F = 0$ which are uniformly continuous and have uniformly at most one root.

Since $f_F = F$ whenever $F \in \widehat{S}$, the mapping

$$\Phi(S) \to \widehat{S}\,,\; F \mapsto f_F$$

has a cross section. The relation $\approx$ on $\Phi(S)$ defined by

$$F \approx G \;\Leftrightarrow\; \forall\delta\,\exists\varepsilon\,\forall x, y\,[F(x) < \varepsilon \wedge G(y) < \varepsilon \;\Rightarrow\; d(x, y) < \delta]$$

is an equivalence relation, and satisfies

$$F \approx G \Leftrightarrow f_F = f_G\,.$$

In all, $F \mapsto f_F$ induces a bijection

$$\Phi(S)\,/\approx \;\leftrightarrow\; \widehat{S}\;.$$

Also, the following are equivalent: $F \approx \widehat{y}$, $f_F = \widehat{y}$, and $F(y) = 0$.